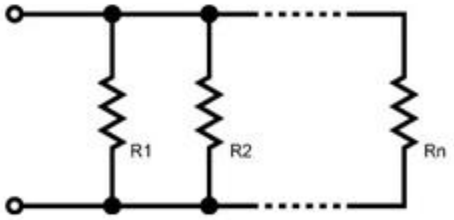
Multivariate Calculus/Mathematics IV
Mid Term Examination Fall Semester- 2022

When two resistors having resistances $R1$ohms and $R2$ohms are connected in parallel, their combined resistance R in ohms is $R = R1R2\ R1+R2$. Show that $\partial\ 2R\ \partial\ 2R\ \partial R1\ 2\ \partial R2\ 2 = 4R\ 2$ $(R1+\ R2)\ 4$



Both resistors are subjected to the same voltage U = U1 = U2
The intensity of the generator current is equal to the sum of the intensities of the currents flowing in the resistors:
I = I1+ I2
Ohm's law applied to each of the resistors gives
U1 = R1 I1
U2 = R2 I 2

$$I_1 = \frac{U}{R_1}$$

$$I_2 = \frac{U}{R_2}$$

$$I = I_1 + I_2 = \frac{U}{R_1} + \frac{U}{R_2} = U(\frac{1}{R_1} + \frac{1}{R_2})$$

We can deduce the equivalent conductance 1 / R

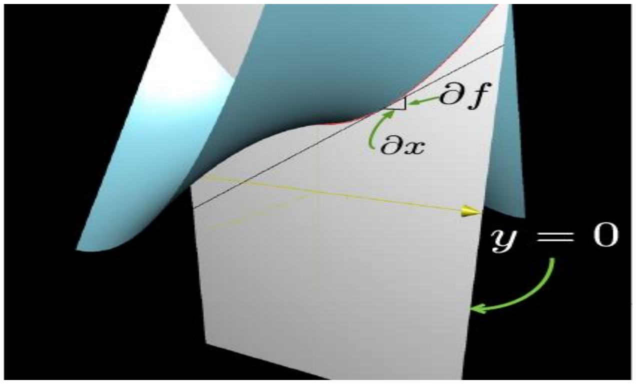
$$\frac{I}{U} = \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

The conductance of a set of resistors in parallel is equal to the sum of their conductances
In the case of 2 resistances the relation can be put in the form:

$$R = \frac{R_1 R_2}{R_1 + R_2}$$

QUESTION#02: (06) Let $u(w, x, y, z) = xe\ ywsin2\ z$ Find i. $\partial\ 4u\ \partial x\partial y\partial w\partial z$ ii. $\partial\ 4u\ \partial w\partial z\partial y2$

- For a multivariable function, like f(x, y) = x^2y*f*(*x*,*y*)=*x*2*y*f, left parenthesis, x, comma, y, right parenthesis, equals, x, squared, y, computing partial derivatives looks something like this: \begin{aligned} \dfrac{\partial f}{\blueE{\partial x}} &= \!\!\!\!\! \underbrace{ \dfrac{\partial}{\blueE{\partial x}} \blueE{x^2}y }_{\substack{ \text{Treat }y\text{ as constant;}\\\\ \text{take derivative.} }}\!\!\!\!\! = 2\blueE{x}y \\\\ \dfrac{\partial f}{\redE{\partial y}} &= \!\!\!\!\! \underbrace{ \dfrac{\partial}{\redE{\partial y}} x^2\redE{y} }_{\substack{ \text{Treat }x\text{ as constant;}\\\\ \text{take derivative.} }}\!\!\!\!\! = x^2\cdot \redE{1} \end{aligned}*∂x∂f∂y∂f*=Treat *y* as constant;take derivative.*∂x∂x*2*y* =2*xy*=Treat *x* as constant;take derivative.*∂y∂x*2*y*=*x*2⋅1
- This swirly-d symbol, \partial∂\partial, often called "del", is used to distinguish partial derivatives from ordinary single-variable derivatives. Or, should I say ... to differentiate them.
- The reason for a new type of derivative is that when the input of a function is made up of multiple variables, we want to see how the function changes as we *let just one of those variables change* while holding all the others constant.
- With respect to three-dimensional graphs, you can picture the partial derivative \dfrac{\partial f}{\partial x}*∂x∂f*start fraction, \partial, f, divided by, \partial, x, end fraction by slicing the graph of f*f*f with a plane representing a constant y*y*y-value and measuring the slope of the resulting curve along the cut.



Intersecting y=0 plane with the graph

What is a *partial* derivative?

We'll assume you are familiar with the ordinary derivative \dfrac{df}{dx}*dxdf*start fraction, d, f, divided by, d, x, end fraction from single variable calculus. I actually quite like this notation for the derivative, because you can interpret it as follows:

- Interpret dx*dx*d, x as "a very tiny change in x*x*x".
- Interpret df*df*d, f as "a very tiny change in the output of f*f*f", where it is understood that this tiny change is whatever results from the tiny change dx*dx*d, x to the input.

  In fact, I think this intuitive feel for the symbol \dfrac{df}{dx}*dxdf*start fraction, d, f, divided by, d, x, end fraction is one of the most useful takeaways from single-variable calculus, and when you really start feeling it in your bones, most of the concepts around derivatives start to click.

  For example, when you apply it to the graph of f*f*f, you can interpret this "ratio" \dfrac{df}{dx}*dxdf*start fraction, d, f, divided by, d, x, end fraction as the rise-over-run slope of the graph of f*f*f, which depends on the point where you started.

y*y*x*x*\large dx*dx*\large df*df*\Large
\underbrace{\quad\;\quad\;\quad\;\quad\;\quad\;\quad\;\quad\;\quad\;}\Large
\substack{\text{Input can only change} \\ \text{along one
axis}}Input can only changealong one axis
Interpretation of\\dfrac{df}{dx}*dfracdfdx* in a single variable function.

How does this work for multivariable functions?

Consider some function with a two-dimensional input and a one-dimensional output.

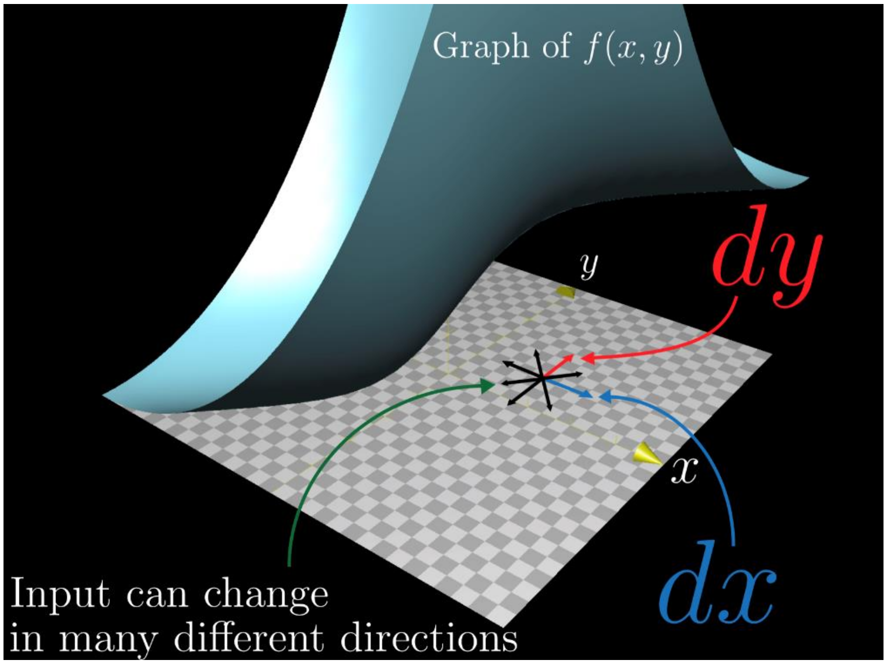
f(x, y) = x^2-2xy*f*(*x*,*y*)=*x*2−2*xy*f, left parenthesis, x, comma, y, right parenthesis, equals, x, squared, minus, 2, x, y

There's nothing stopping us from writing the same expression, \dfrac{df}{dx}*dxdf*start fraction, d, f, divided by, d, x, end fraction, and interpreting it the same way:

- dx*dx*d, x can still represent a tiny change in the variable x*x*x, which is now just one component of our input.
- df*df*d, f can still represent the resulting change to the output of the function f(x, y)*f*(*x*,*y*)f, left parenthesis, x, comma, y, right parenthesis.

  However, this ignores the fact that there is another input variable y*y*y. The input space now has multiple dimensions, so we can change the input in many directions other than the x*x*x-direction.

  For example, what about changing y*y*y slightly by some small value dy*dy*d, y? Now if we re-interpret df*df*d, f to represent the tiny change to the function that this dy*dy*d, y shift brings about, we would have a different derivative \dfrac{df}{dy}*dydf*start fraction, d, f, divided by, d, y, end fraction.



Indication that the input of a multivariable function can change in many directions.

Neither one of these derivatives tells the full story of how our function f(x, y)*f*(*x*,*y*)f, left parenthesis, x, comma, y, right parenthesis changes when its input changes slightly, so we call them **partial derivatives**. To emphasize the difference, *we no longer use the letter ddd to indicate tiny changes, but instead introduce a newfangled symbol \partial∂\partial* to do the trick, writing each partial derivative as \dfrac{\partial f}{\partial x}*∂x∂f*start fraction, \partial, f, divided by, \partial, x, end fraction, \dfrac{\partial f}{\partial y}*∂y∂f*start fraction, \partial, f, divided by, \partial, y, end fraction, etc.

You read the symbol \dfrac{\partial f}{\partial x}*∂x∂f*start fraction, \partial, f, divided by, \partial, x, end fraction out loud by saying "the partial derivative of f*f*f with respect to x*x*x".

\begin{array}{c} \text{Can be thought of as } \\ \text{"a tiny change in} \\ \text{the function's output"} \\\\ \begin{array}{c} \begin{array}{c} \blueE{\text{Used instead of "d" in}} \\ \blueE{\text{usual }\dfrac{df}{dx}\text{ notation to}} \\ \blueE{\text{emphasize that this is}} \\ \blueE{\text{a partial derivative.}} \end{array} & \begin{array}{c} \LARGE\blueE\nearrow^{\huge\,\overbrace{\blueE\partial \greenE f}\,}\greenE \nwarrow \\\\ \Huge \text{\textemdash} \\\\ \LARGE\blueE\searrow_{\huge\,\underbrace{\blueE\partial \maroonE x}\,}\maroonE \swarrow \end{array} & \begin{array}{c} \\\\ \greenE{\text{Multivariable}} \\ \greenE{\text{function}} \\\\ \maroonE{\text{Indicates which}} \\ \maroonE{\text{input variable}} \\ \maroonE{\text{is changed slightly.}} \end{array} \end{array} \\\\ \text{Can be thought of as} \\ \text{"a tiny change in }x\text{"} \end{array}Can be thought of as "a tiny change inthe function's output"Used instead of "d" inusual *dxdf* notation toemphasize that this isa partial derivative.↗*∂f*↖—↘*∂x*↙ MultivariablefunctionIndicates whichinput variableis changed slightly.
Can be thought of as"a tiny change in *x*"

Example: Computing a partial derivative

Consider this function:

f(\blueE{x}, \redE{y}) = \blueE{x}^2 \redE{y}^3f(x,y)=x2y3f, left parenthesis, start color #0c7f99, x, end color #0c7f99, comma, start color #bc2612, y, end color #bc2612, right parenthesis, equals, start color #0c7f99, x, end color #0c7f99, squared, start color #bc2612, y, end color #bc2612, cubed

Suppose I asked you to evaluate \dfrac{\partial f}{\blueE{\partial x}}∂x∂fstart fraction, \partial, f, divided by, start color #0c7f99, \partial, x, end color #0c7f99, end fraction, the partial derivative with respect to xxx, at the input (\blueE{3}, \redE{2})(3,2)left parenthesis, start color #0c7f99, 3, end color #0c7f99, comma, start color #bc2612, 2, end color #bc2612, right parenthesis.

"What? But I haven't learned how yet!"

Don't worry, it's mostly just the same mechanics as an ordinary derivative.

From the introduction above, you should know that this is asking about the rate at which the output of fff changes as we nudge the xxx-component of the input slightly, perhaps moving from (\blueE{3}, \redE{2})(3,2)left parenthesis, start color #0c7f99, 3, end color #0c7f99, comma, start color #bc2612, 2, end color #bc2612, right parenthesis to (\blueE{3.01}, \redE{2})(3.01,2)left parenthesis, start color #0c7f99, 3, point, 01, end color #0c7f99, comma, start color #bc2612, 2, end color #bc2612, right parenthesis.

Since we only care about movement in the \blueE{x}xstart color #0c7f99, x, end color #0c7f99-direction, we might as well treat the \redE{y}ystart color #bc2612, y, end color #bc2612-value as a constant. In fact, we can just plug in \redE{y=2}y=2start color #bc2612, y, equals, 2, end color #bc2612 ahead of time before computing any derivatives:

f(\blueE{x}, \redE{2}) = \blueE{x}^2 (\redE{2})^3 = 8\blueE{x}^2f(x,2)=x2(2)3=8x2f, left parenthesis, start color #0c7f99, x, end color #0c7f99, comma, start color #bc2612, 2, end color #bc2612, right parenthesis, equals, start color #0c7f99, x, end color #0c7f99, squared, left parenthesis, start color #bc2612, 2, end color #bc2612, right parenthesis, cubed, equals, 8, start color #0c7f99, x, end color #0c7f99, squared

Now, asking how fff changes in response to a small shift in \blueE{x}xstart color #0c7f99, x, end color #0c7f99 is just an ordinary, single-variable derivative.

CONCEPT CHECK

What is the derivative of this function f(\blueE{x}, \redE{2}) = 8\blueE{x}^2f(x,2)=8x2f, left parenthesis, start color #0c7f99, x, end color #0c7f99, comma, start color #bc2612, 2, end color #bc2612, right parenthesis, equals, 8, start color #0c7f99, x, end color #0c7f99, squared evaluated at \blueE{x = 3}x=3start color #0c7f99, x, equals, 3, end color #0c7f99?
Without pre-evaluating yyy

Now suppose I asked you to find \dfrac{\partial f}{\blueE{\partial x}}∂x∂fstart fraction, \partial, f, divided by, start color #0c7f99, \partial, x, end color #0c7f99, end fraction, but I didn't ask you to evaluate it at a specific point. In other words, you should give me new multivariable function which takes *any* point (\blueE{x}, \redE{y})(x,y)left parenthesis, start color #0c7f99, x, end color #0c7f99, comma, start color #bc2612, y, end color #bc2612, right parenthesis as its input and tells me what the rate of change of fff near that point is as we move purely in the \blueE{x}xstart color #0c7f99, x, end color #0c7f99-direction.

You can start the same way, treating the \redE{y}ystart color #bc2612, y, end color #bc2612 value as a constant. However, this time, you cannot plug in an actual constant value, like \redE{y = 2}y=2start color #bc2612, y, equals, 2, end color #bc2612.

Instead, *pretend* that \redE{y}ystart color #bc2612, y, end color #bc2612 is constant and take the derivative:

\dfrac{d}{\blueE{dx}}f(\blueE{x}, y) = \underbrace{ \dfrac{d}{\blueE{dx}}(\blueE{x}^2y^3) }_{\text{Pretend }y\text{ is constant}} = 2\blueE{x}y^3dxdf(x,y)=Pretend y is constantdxd(x2y3)=2xy3start fraction, d, divided by, start color #0c7f99, d, x, end color #0c7f99, end fraction, f, left parenthesis, start color #0c7f99, x, end color #0c7f99, comma, y, right parenthesis, equals, start underbrace, start fraction, d, divided by, start color #0c7f99, d, x, end color #0c7f99, end fraction, left parenthesis, start color #0c7f99, x, end color #0c7f99, squared, y, cubed, right parenthesis, end underbrace, start subscript, start text, P, r, e, t, e, n, d, space, end text, y, start text, space, i, s, space, c, o, n, s, t, a, n, t, end text, end subscript, equals, 2, start color #0c7f99, x, end color #0c7f99, y, cubed

Or rather, since to emphasize that this is a multivariable function, we use the symbol \partial∂\partial instead of ddd:

\dfrac{\partial}{\blueE{\partial x}}f(\blueE{x}, y) = \dfrac{\partial}{\blueE{\partial x}}(\blueE{x}^2y^3) = 2\blueE{x}y^3∂x∂f(x,y)=∂x∂(x2y3)=2xy3start fraction, \partial, divided by, start color #0c7f99, \partial, x, end color #0c7f99, end fraction, f, left parenthesis, start color #0c7f99, x, end color #0c7f99, comma, y, right parenthesis, equals, start fraction, \partial, divided by, start color #0c7f99, \partial, x, end color #0c7f99, end fraction, left parenthesis, start color #0c7f99, x, end color #0c7f99, squared, y, cubed, right parenthesis, equals, 2, start color #0c7f99, x, end color #0c7f99, y, cubed

As a sanity check, you can plug in (\blueE{3}, \redE{2})(3,2)left parenthesis, start color #0c7f99, 3, end color #0c7f99, comma, start color #bc2612, 2, end color #bc2612, right parenthesis to see that we get the same result as above.

"So, what's the difference between \dfrac{d}{dx}dxdstart fraction, d, divided by, d, x, end fraction and \dfrac{\partial}{\partial x}∂x∂start fraction, \partial, divided by, \partial, x, end fraction? They seem to be used the same way."

Honestly, as far as I'm concerned, there's not really a difference between these operations. You could be pedantic and say one is only defined for single variable functions. But as far as intuition and computation go, they are one and the same, and the difference is just meant to clarify what *type* of function is being differentiated.

Interpreting partial derivatives with graphs

Consider this function:

f(x, y) = \dfrac{1}{5}(x^2 - 2xy) + 3f(x,y)=51(x2−2xy)+3f, left parenthesis, x, comma, y, right parenthesis, equals, start fraction, 1, divided by, 5, end fraction, left parenthesis, x, squared, minus, 2, x, y, right parenthesis, plus, 3,

Here is a video showing its graph rotating, just to get a feel for the three-dimensional nature of it.

Khan Academy video wrapper

See video transcript

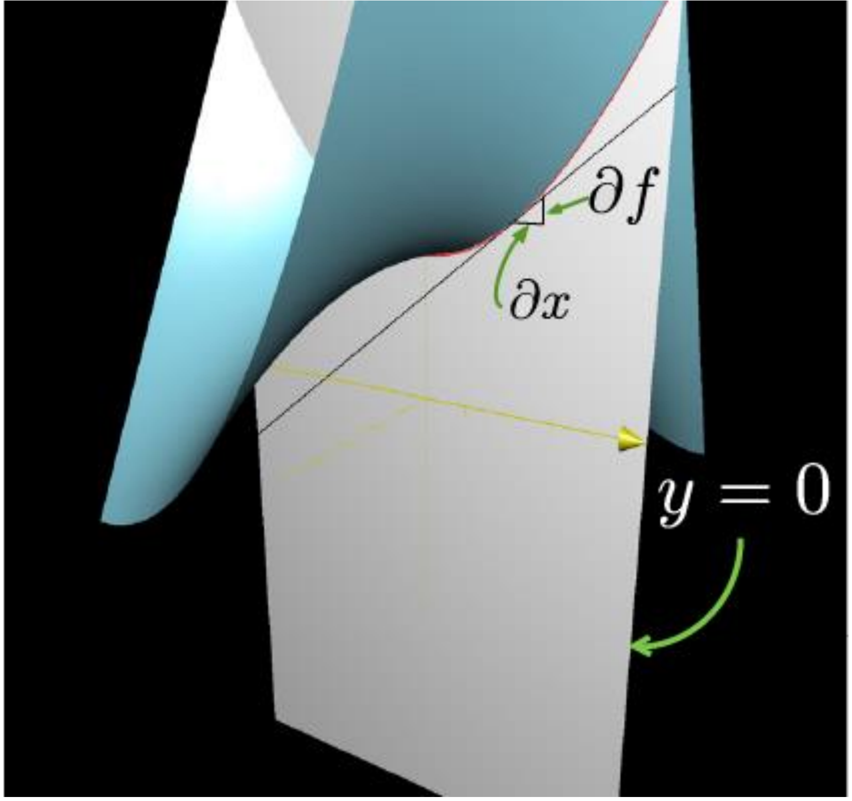
Think about the partial derivative of fff with respect to \blueE{x}xstart color #0c7f99, x, end color #0c7f99, perhaps evaluated at the point (2, 0)(2,0)left parenthesis, 2, comma, 0, right parenthesis.

\dfrac{\partial f}{\blueE{\partial x}}(2, 0)∂x∂f(2,0)start fraction, \partial, f, divided by, start color #0c7f99, \partial, x, end color #0c7f99, end fraction, left parenthesis, 2, comma, 0, right parenthesis

In terms of the graph, what does the value of this expression tell us about the behavior of the function fff at the point (2, 0)(2,0)left parenthesis, 2, comma, 0, right parenthesis?

Treat y*y*y as constant \rightarrow→right arrow slice graph with plane

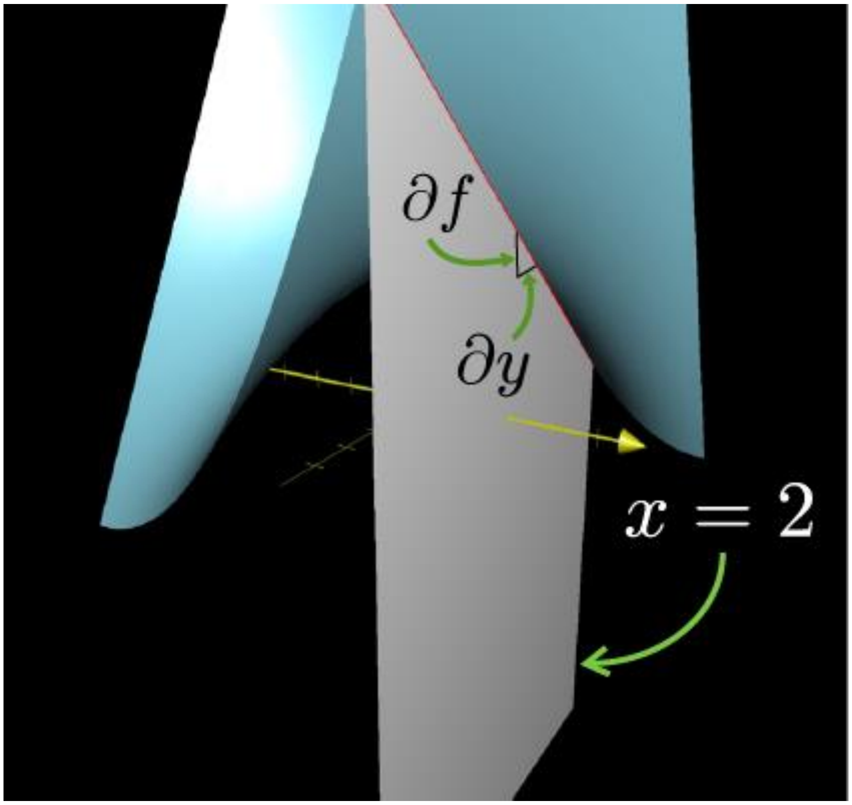
The first step when computing this value is to treat y*y*y as a constant. Specifically, if we are limiting our view to what happens at the point (2, 0)(2,0)left parenthesis, 2, comma, 0, right parenthesis, we should only look at the set of points where y = 0*y*=0y, equals, 0. In three-dimensional space, this set is plane perpendicular to the y*y*y-axis, passing through the origin.



Intersecting y=0 plane with the graph

This plane y = 0*y*=0y, equals, 0, shown in white, slices into the graph of f(x,y)*f*(*x*,*y*)f, left parenthesis, x, comma, y, right parenthesis along a parabolic curve, shown faintly in red. We can interpret \dfrac{\partial f}{\blueE{\partial x}}*∂x∂f*start fraction, \partial, f, divided by, start color #0c7f99, \partial, x, end color #0c7f99, end fraction as giving the slope of a tangent line to this curve. Why? Because \partial x*∂x*\partial, x is a slight nudge in the x*x*x-direction, the run, and \partial f*∂f*\partial, f is the resulting change in the z*z*z-direction, the rise.

What about \dfrac{\partial f}{\redE{\partial y}}*∂y∂f*start fraction, \partial, f, divided by, start color #bc2612, \partial, y, end color #bc2612, end fraction at that same point (2, 0)(2,0)left parenthesis, 2, comma, 0, right parenthesis? The points where \blueE{x=2}*x*=2start color #0c7f99, x, equals, 2, end color #0c7f99 also make up a plane, but this time it's a plane perpendicular to the x*x*x-axis intersecting the point x=2*x*=2x, equals, 2. This slices the graph along a new curve, and \dfrac{\partial f}{\redE{\partial y}}*∂y∂f*start fraction, \partial, f, divided by, start color #bc2612, \partial, y, end color #bc2612, end fraction will give the slope of that new curve.



Intersecting x=2 plane with the graph.

REFLECTION QUESTION

In the picture to the right, the "curve" where the graph of f(x, y) = \dfrac{1}{5}(x^2 - 2xy) + 3*f*(*x*,*y*)=51(*x*2−2*xy*)+3f, left parenthesis, x, comma, y, right parenthesis, equals, start fraction, 1, divided by, 5, end fraction, left parenthesis, x, squared, minus, 2, x, y, right parenthesis, plus, 3 intersects the plane defined by x=2*x*=2x, equals, 2 looks like it might be a straight line.

Is it really a line?

Choose 1 answer:

Choose 1 answer:

- ○
  (Choice A)
  A
  Yes
- ○
  (Choice B)
  B
  No

CheckExplain

[Deeper evaluation of this partial derivative]

Phrasing and notation

Here are some of the phrases you might hear in reference to this \dfrac{\partial f}{\partial x}*∂x∂f* start fraction, \partial, f, divided by, \partial, x, end fraction operation:

- "The partial derivative of f*f*f with respect to x*x*x"
- "Del f, del x"
- "Partial f, partial x"
- "The partial derivative (of f*f*f) in the x*x*x-direction"

Alternate notation

In the same way that people sometimes prefer to write f'*f′*f, prime instead of \dfrac{df}{dx}*dxdf* start fraction, d, f, divided by, d, x, end fraction, we have the following notation:
\begin{aligned} f_\blueE{x} &\leftrightarrow \dfrac{\partial f}{\blueE{\partial x}} \\\\ f_\redE{y} &\leftrightarrow \dfrac{\partial f}{\redE{\partial y}} \\\\ f_{\greenE{\langle\text{Some variable }\rangle}} &\leftrightarrow \dfrac{\partial f}{\greenE{\partial \langle\text{That same variable} \rangle}} \end{aligned}*fxfy*
f⟨Some variable ⟩↔∂*x*∂*f*↔∂*y*∂*f*↔∂⟨That same variable⟩∂*f*

A note about "del"

While it's common to refer to the partial symbol \partial∂\partial as "del", this can be confusing because "del" is also the name of the Nabla symbol \nabla∇del, which we will introduce in the next article.

A more formal definition

Although thinking of dx*dx*d, x or \partial x∂*x*\partial, x as really tiny changes in the value of x*x*x is a useful intuition, it is healthy to occasionally step back and remember that defining things precisely requires introducing limits. After all, what specific small value would \partial x∂*x*\partial, x be? One one hundredth? One one millionth? 10^{-10^{10}}10−101010, start superscript, minus, 10, start superscript, 10, end superscript, end superscript?

The point of calculus is that we don't use any one tiny number, but instead consider all possible values and analyze what tends to happen as they approach a limiting value. The single variable derivative, for example, is defined like this:
\begin{aligned} \dfrac{{df}}{\blueE{dx}}(x_0) = \lim_{\blueE{h}\to 0} \dfrac{{f(x_0\blueE{+h}) - f(x_0)}}{\blueE{h}} \end{aligned}*dxdf*(*x*0)=*h*→0lim*hf*(*x*0+*h*)−*f*(*x*0)

- h*h*h represents the "tiny value" that we intuitively think of as dx*dx*d, x.
- The h \to 0*h*→0h, \to, 0 under the limit indicates that we care about very small values of h*h*h, those approaching 000.
- f(x_0 + h) - f(x_0)*f*(*x*0+*h*)−*f*(*x*0)f, left parenthesis, x, start subscript, 0, end subscript, plus, h, right parenthesis, minus, f, left parenthesis, x, start subscript, 0, end subscript, right parenthesis is the change in the output that results from adding h*h*h to the input, which is what we think of as df*df*d, f.

Formally defining the partial derivative looks almost identical. If f(x, y, \dots)*f*(*x*,*y*,…)f, left parenthesis, x, comma, y, comma, dots, right parenthesis is a function with multiple inputs, here's how that looks:
\begin{aligned} \dfrac{\partial f}{\blueE{\partial x}}(x_0, y_0, \dots) &= \lim_{\blueE{h} \to 0} \dfrac{f(\blueE{x_0\blueE{+h}}, y_0, \dots) - f(x_0, y_0, \dots)} {\blueE{h}} \end{aligned}∂*x*∂*f*(*x*0,*y*0,…)=*h*→0lim*hf*(*x*0+*h*,*y*0,…)−*f*(*x*0,*y*0,…)

Similarly, here's how the partial derivative with respect to y*y*y looks:
\begin{aligned} \dfrac{\partial f}{\redD{\partial y}}(x_0, y_0, \dots) &= \lim_{\redD{h} \to 0} \dfrac{f(x_0, \redD{y_0+h}, \dots) - f(x_0, y_0, \dots)}{\redD{h}} \end{aligned}∂*y*∂*f*(*x*0,*y*0,…) =*h*→0lim*hf*(*x*0,*y*0+*h*,…)−*f*(*x*0,*y*0,…)
The point is that h*h*h, which represents a tiny tweak to the input, is added to different input variables depending on which partial derivative we are taking.
People will often refer to this as the **limit definition** of a partial derivative.
Reflection question: How can we think about this limit definition in the context of the graphical interpretation above? What is h*h*h? What does it look like for h \to 0*h*→0h, \to, 0?

Summary

- For a multivariable function, like f(x, y) = x^2y*f*(*x*,*y*)=*x*2*y*f, left parenthesis, x, comma, y, right parenthesis, equals, x, squared, y, computing partial derivatives looks something like this: \begin{aligned} \dfrac{\partial f}{\blueE{\partial x}} &= \!\!\!\!\! \underbrace{ \dfrac{\partial}{\blueE{\partial x}} \blueE{x^2}y }_{\substack{ \text{Treat }y\text{ as constant;}\\\\ \text{take derivative.} }}\!\!\!\!\! = 2\blueE{x}y \\\\ \dfrac{\partial f}{\redE{\partial y}} &= \!\!\!\!\! \underbrace{ \dfrac{\partial}{\redE{\partial y}} x^2\redE{y} }_{\substack{ \text{Treat }x\text{ as constant;}\\\\ \text{take derivative.} }}\!\!\!\!\! = x^2\cdot \redE{1} \end{aligned}∂*x*∂*f*∂*y*∂*f*=Treat *y* as constant;take derivative.∂*x*∂*x*2*y* =2*xy*=Treat *x* as constant;take derivative.∂*y*∂*x*2*y*=*x*2⋅1
- This swirly-d symbol \partial∂\partial, often called "del", is used to distinguish partial derivatives from ordinary single-variable derivatives.
- The reason for a new type of derivative is that when the input of a function is made up of multiple variables, we want to see how the function changes as we *let just one of those variables change* while holding all the others constant.
- With respect to three-dimensional graphs, you can picture the partial derivative \dfrac{\partial f}{\partial x}∂*x*∂*f*start fraction, \partial, f, divided by, \partial, x, end fraction by slicing the graph of f*f*f with a plane representing a constant y*y*y-value, and measuring the slope of the resulting cut.

QUESTION#03: (06) Express *∂w/∂u and ∂w/∂v* as functions of u and v both by using the Chain Rule and expressing w directly in terms of u and v before differentiating. *w* = ln(*x* 2 + *y* 2 + *z* 2) , *x* = *e uv* sin *u*, *y* = *e v* cos *vu* , *z* = *uve v* : (*u* , *v*) = (−2, 0)

Recall that the chain rule for the derivative of a composite of two functions can be written in the form

ddx(f(g(x)))=f'(g(x))g'(x).ddx(f(g(x)))=f′(g(x))g′(x).

In this equation, both f(x)f(x) and g(x)g(x) are functions of one variable. Now suppose that ff is a function of two variables and gg is a function of one variable. Or perhaps they are both functions of two variables, or even more. How would we calculate the derivative in these cases? The following theorem gives us the answer for the case of one independent variable.

THEOREM 4.8

Chain Rule for One Independent Variable

Suppose that x=g(t)x=g(t) and y=h(t)y=h(t) are differentiable functions of tt and z=f(x,y)z=f(x,y) is a differentiable function of xandy.xandy. Then z=f(x(t),y(t))z=f(x(t),y(t)) is a differentiable function of tt and

dzdt=∂z∂x·dxdt+∂z∂y·dydt,dzdt=∂z∂x·dxdt+∂z∂y·dydt,

(4.29)

where the ordinary derivatives are evaluated at tt and the partial derivatives are evaluated at (x,y).(x,y).

Proof

The proof of this theorem uses the definition of differentiability of a function of two variables. Suppose that *f* is differentiable at the point P(x0,y0),P(x0,y0), where x0=g(t0)x0=g(t0) and y0=h(t0)y0=h(t0) for a fixed value of t0.t0. We wish to prove that z=f(x(t),y(t))z=f(x(t),y(t)) is differentiable at t=t0t=t0 and that Equation 4.29 holds at that point as well.

Since ff is differentiable at P,P, we know that

z(t)=f(x,y)=f(x0,y0)+fx(x0,y0)(x−x0)+fy(x0,y0)(y−y0)+E(x,y),z(t)=f(x,y)=f(x0,y0)+fx(x0,y0)(x −x0)+fy(x0,y0)(y−y0)+E(x,y),

(4.30)

where lim(x,y)→(x0,y0)E(x,y)(x−x0)2+(y−y0)2√=0.lim(x,y)→(x0,y0)E(x,y)(x−x0)2+(y−y0)2= 0. We then subtract z0=f(x0,y0)z0=f(x0,y0) from both sides of this equation:

z(t)−z(t0)=f(x(t),y(t))−f(x(t0),y(t0))=fx(x0,y0)(x(t)−x(t0))+fy(x0,y0)(y(t)−y(t0))+E(x(t),y(t)).z(t) −z(t0)=f(x(t),y(t))−f(x(t0),y(t0))=fx(x0,y0)(x(t)−x(t0))+fy(x0,y0)(y(t)−y(t0))+E(x(t),y(t)).

Next, we divide both sides by t−t0:t−t0:

z(t)−z(t0)t−t0=fx(x0,y0)(x(t)−x(t0)t−t0)+fy(x0,y0)(y(t)−y(t0)t−t0)+E(x(t),y(t))t−t0.z(t)−z(t0)t−t0 =fx(x0,y0)(x(t)−x(t0)t−t0)+fy(x0,y0)(y(t)−y(t0)t−t0)+E(x(t),y(t))t−t0.

Then we take the limit as tt approaches t0:t0:

limt→t0z(t)−z(t0)t−t0=fx(x0,y0)limt→t0(x(t)−x(t0)t−t0)+fy(x0,y0)limt→t0(y(t)−y(t0)t−t0)+limt →t0E(x(t),y(t))t−t0.limt→t0z(t)−z(t0)t−t0=fx(x0,y0)limt→t0(x(t)−x(t0)t−t0)+fy(x0,y0)limt→t0(y(t)−y(t0)t−t0)+limt→t0E(x(t),y(t))t−t0.

The left-hand side of this equation is equal to dz/dt,dz/dt, which leads to

dzdt=fx(x0,y0)dxdt+fy(x0,y0)dydt+limt→t0E(x(t),y(t))t−t0.dzdt=fx(x0,y0)dxdt+fy(x0,y0)dydt+limt→t0E(x(t),y(t))t−t0.

The last term can be rewritten as

limt→t0E(x(t),y(t))t−t0=limt→t0(E(x,y)(x−x0)2+(y−y0)2√(x−x0)2+(y−y0)2√t−t0)=limt→t0(E(x,y)(x−x0)2+(y−y0)2√)limt→t0((x−x0)2+(y−y0)2√t−t0).limt→t0E(x(t),y(t))t−t0=limt→t0(E(x,y)(x−x0)2+(y−y0)2(x−x0)2+(y−y0)2t−t0)=limt→t0(E(x,y)(x−x0)2+(y−y0)2)limt→t0((x−x0)2+(y−y0)2t−t0).

As tt approaches t0,t0, (x(t),y(t))(x(t),y(t)) approaches (x(t0),y(t0)),(x(t0),y(t0)), so we can rewrite the last product as

lim(x,y)→(x0,y0)⎛⎝⎜ E(x,y)(x−x0)2+(y−y0)2−−−−−−−−−−−−−−−−−√ ⎞⎠⎟ ⎛⎝⎜ (x−x0)2+(y−y0)2−−−−−−−−−−−−−−−−−√t−t0 ⎞⎠⎟=lim(x,y)→(x0,y0).lim(x,y)→(x0,y0)(E(x,y)(x−x0)2+(y−y0)2)=lim(x,y)→(x0,y0)((x−x0)2+(y−y0)2t−t0).

Since the first limit is equal to zero, we need only show that the second limit is finite:

lim(x,y)→(x0,y0)((x−x0)2+(y−y0)2√t−t0)=lim(x,y)→(x0,y0)((x−x0)2+(y−y0)2(t−t0)2−−−−−−−−−−−√)=lim(x,y)→(x0,y0)((x−x0t−t0)2+(y−y0t−t0)2−−−−−−−−−−−−−−−−√)=(lim(x,y)→(x0,y0)(x−x0t−t0))2+(lim(x,y)→(x0,y0)(y−y0t−t0))2−−−−−−−−−−−−−−−−−−−−−−−−−−−−−−−−−−−−−√.lim(x,y)→(x0,y0)((x−x0)2+(y−y0)2t−t0)=lim(x,y)→(x0,y0)((x−x0)2+(y−y0)2(t−t0)2)=lim(x,y)→(x0,y0)((x−x0t−t0)2+(y−y0t−t0)2)=(lim(x,y)→(x0,y0)(x−x0t−t0))2+(lim(x,y)→(x0,y0)(y−y0t−t0))2.

Since x(t)x(t) and y(t)y(t) are both differentiable functions of t,t, both limits inside the last radical exist. Therefore, this value is finite. This proves the chain rule at t=t0;t=t0; the rest of the theorem follows from the assumption that all functions are differentiable over their entire domains.

□

Closer examination of Equation 4.29 reveals an interesting pattern. The first term in the equation is ∂f∂x·dxdt∂f∂x·dxdt and the second term is ∂f∂y·dydt.∂f∂y·dydt. Recall that when multiplying fractions, cancelation can be used. If we treat these derivatives as fractions, then each product “simplifies” to something resembling ∂f/dt.∂f/dt. The variables xandyxandy that disappear in this simplification are often called **intermediate variables**: they are independent variables for the function f,f, but are dependent variables for the variable t.t. Two terms appear on the right-hand side of the formula, and ff is a function of two variables. This pattern works with functions of more than two variables as well, as we see later in this section.

EXAMPLE 4.26

Using the Chain Rule

Calculate dz/dtdz/dt for each of the following functions:

a. z=f(x,y)=4x2+3y2,x=x(t)=sint,y=y(t)=costz=f(x,y)=4x2+3y2,x=x(t)=sint,y=y(t)=cost
b. z=f(x,y)=x2−y2−−−−−−√,x=x(t)=e2t,y=y(t)=e−tz=f(x,y)=x2−y2,x=x(t)=e2t,y=y(t)

CHECKPOINT 4.23

Calculate dz/dtdz/dt given the following functions. Express the final answer in terms of t.t.

z=f(x,y)=x2−3xy+2y2,x=x(t)=3sin2t,y=y(t)=4cos2tz=f(x,y)=x2−3xy+2y2,x=x(t)=3sin2t,y=y(t)=4cos2t

It is often useful to create a visual representation of Equation 4.29 for the chain rule. This is called a **tree diagram** for the chain rule for functions of one variable and it provides a way to remember the formula (Figure 4.34). This diagram can be expanded for functions of more than one variable, as we shall see very shortly.

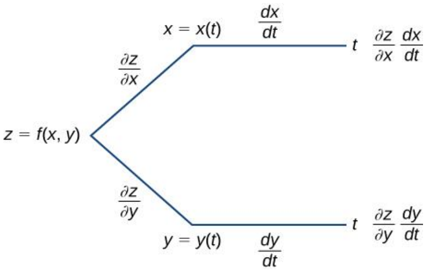


Figure 4.34 Tree diagram for the case dzdt=∂z∂x·dxdt+∂z∂y·dydt.dzdt=∂z∂x·dxdt+∂z∂y·dydt.

In this diagram, the leftmost corner corresponds to z=f(x,y).z=f(x,y). Since ff has two independent variables, there are two lines coming from this corner. The upper branch corresponds to the variable xx and the lower branch corresponds to the variable y.y. Since each of these variables is then dependent on one variable t,t, one branch then comes from xx and one branch comes from y.y. Last, each of the branches on the far right has a label that represents the path traveled to reach that branch. The top branch is reached by following the xx branch, then the tt branch; therefore, it is labeled (∂z/∂x)×(dx/dt).(∂z/∂x)×(dx/dt). The bottom branch is similar: first the yy branch, then the tt branch. This branch is labeled (∂z/∂y)×(dy/dt).(∂z/∂y)×(dy/dt). To get the formula for dz/dt,dz/dt, add all the terms that appear on the rightmost side of the diagram. This gives us Equation 4.29.

In Chain Rule for Two Independent Variables, z=f(x,y)z=f(x,y) is a function of xandy,xandy, and both x=g(u,v)x=g(u,v) and y=h(u,v)y=h(u,v) are functions of the independent variables uandv.uandv.

THEOREM 4.9

Chain Rule for Two Independent Variables

Suppose x=g(u,v)x=g(u,v) and y=h(u,v)y=h(u,v) are differentiable functions of uu and v,v, and z=f(x,y)z=f(x,y) is a differentiable function of xandy.xandy. Then, z=f(g(u,v),h(u,v))z=f(g(u,v),h(u,v)) is a differentiable function of uandv,uandv, and

∂z∂u=∂z∂x∂x∂u+∂z∂y∂y∂u∂z∂u=∂z∂x∂x∂u+∂z∂y∂y∂u

(4.31)

and

∂z∂v=∂z∂x∂x∂v+∂z∂y∂y∂v.∂z∂v=∂z∂x∂x∂v+∂z∂y∂y∂v.

(4.32)

We can draw a tree diagram for each of these formulas as well as follows.

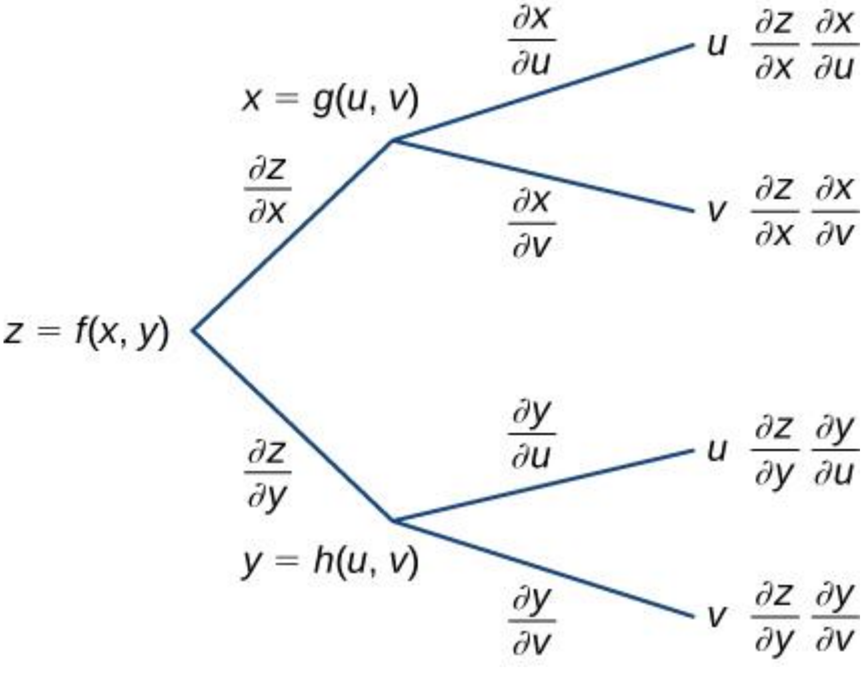


Figure 4.35 Tree diagram for ∂z∂u=∂z∂x·∂x∂u+∂z∂y·∂y∂u∂z∂u=∂z∂x·∂x∂u+∂z∂y·∂y∂u and ∂z∂v=∂z∂x·∂x∂v+∂z∂y·∂y∂v.∂z∂v=∂z∂x·∂x∂v+∂z∂y·∂y∂v.

To derive the formula for ∂z/∂u,∂z/∂u, start from the left side of the diagram, then follow only the branches that end with uu and add the terms that appear at the end of those branches. For the formula for ∂z/∂v,∂z/∂v, follow only the branches that end with vv and add the terms that appear at the end of those branches.

There is an important difference between these two chain rule theorems. In Chain Rule for One Independent Variable, the left-hand side of the formula for the derivative is not a partial derivative, but in Chain Rule for Two Independent Variables it is. The reason is that, in Chain Rule for One Independent Variable, zz is ultimately a function of tt alone, whereas in Chain Rule for Two Independent Variables, zz is a function of both uandv.uandv.

EXAMPLE 4.27

Using the Chain Rule for Two Variables

Calculate ∂z/∂u∂z/∂u and ∂z/∂v∂z/∂v using the following functions:

z=f(x,y)=3x2−2xy+y2,x=x(u,v)=3u+2v,y=y(u,v)=4u−v.z=f(x,y)=3x2−2xy+y2,x=x(u,v)=3u+2v,y=y(u,v)=4u−v.

CHECKPOINT 4.24

Calculate ∂z/∂u∂z/∂u and ∂z/∂v∂z/∂v given the following functions:

z=f(x,y)=2x−yx+3y,x(u,v)=e2ucos3v,y(u,v)=e2usin3v.z=f(x,y)=2x−yx+3y,x(u,v)=e2ucos3v,y(u,v)=e2usin3v.

The Generalized Chain Rule

Now that we’ve see how to extend the original chain rule to functions of two variables, it is natural to ask: Can we extend the rule to more than two variables? The answer is yes, as the **generalized chain rule** states.

THEOREM 4.10

Generalized Chain Rule

Let w=f(x1,x2,…,xm)w=f(x1,x2,…,xm) be a differentiable function of mm independent variables, and for each i∈{1,…,m},i∈{1,…,m}, let xi=xi(t1,t2,…,tn)xi=xi(t1,t2,…,tn) be a differentiable function of nn independent variables. Then

∂w∂tj=∂w∂x1∂x1∂tj+∂w∂x2∂x2∂tj+⋯+∂w∂xm∂xm∂tj∂w∂tj=∂w∂x1∂x1∂tj+∂w∂x2∂x2∂tj+⋯+∂w∂xm∂xm∂tj

(4.33)

for any j∈{1,2,…,n}.j∈{1,2,…,n}.

In the next example we calculate the derivative of a function of three independent variables in which each of the three variables is dependent on two other variables.

EXAMPLE 4.28

Using the Generalized Chain Rule

Calculate ∂w/∂u∂w/∂u and ∂w/∂v∂w/∂v using the following functions:
wxyz====f(x,y,z)=3x2−2xy+4z2x(u,v)=eusinvy(u,v)=eucosvz(u,v)=eu.w=f(x,y,z)=3x2−2xy+4z2x=x(u,v)=eusinvy=y(u,v)=eucosvz=z(u,v)=eu.

CHECKPOINT 4.25

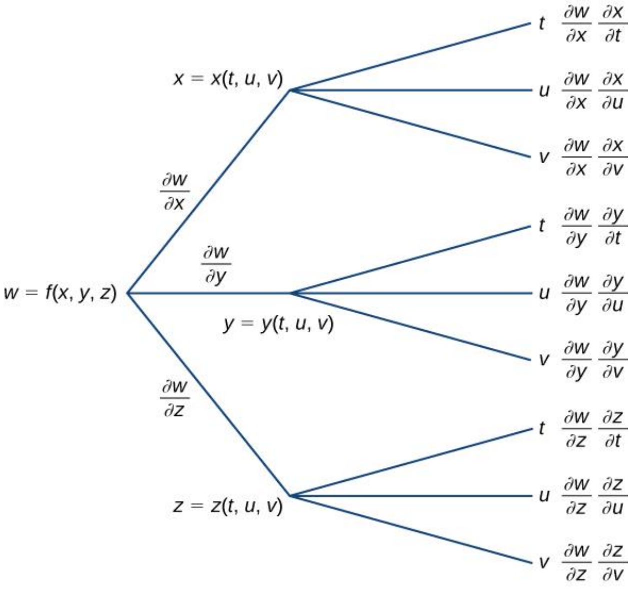
Calculate ∂w/∂u∂w/∂u and ∂w/∂v∂w/∂v given the following functions:
wxyz====f(x,y,z)=x+2y−4z2x−y+3zx(u,v)=e2ucos3vy(u,v)=e2usin3vz(u,v)=e2u.w=f(x,y,z)=x+2y−4z2x−y+3zx=x(u,v)=e2ucos3vy=y(u,v)=e2usin3vz=z(u,v)=e2u.

EXAMPLE 4.29

Drawing a Tree Diagram

Create a tree diagram for the case when
w=f(x,y,z),x=x(t,u,v),y=y(t,u,v),z=z(t,u,v)w=f(x,y,z),x=x(t,u,v),y=y(t,u,v),z=z(t,u,v)

and write out the formulas for the three partial derivatives of w.w.



CHECKPOINT 4.26

Create a tree diagram for the case when
w=f(x,y),x=x(t,u,v),y=y(t,u,v)w=f(x,y),x=x(t,u,v),y=y(t,u,v)
and write out the formulas for the three partial derivatives of w.w.

Implicit Differentiation

Recall from Implicit Differentiation that implicit differentiation provides a method for finding dy/dxdy/dx when yy is defined implicitly as a function of x.x. The method involves differentiating both sides of the equation defining the function with respect to x,x, then solving for dy/dx.dy/dx. Partial derivatives provide an alternative to this method.

Consider the ellipse defined by the equation x2+3y2+4y−4=0x2+3y2+4y−4=0 as follows.

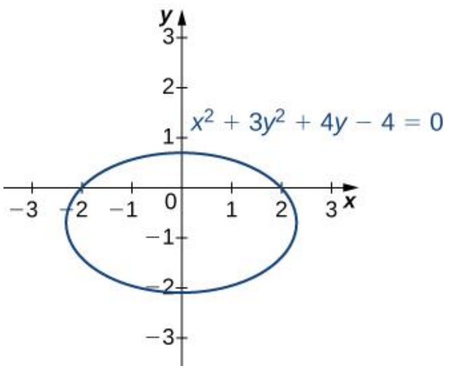


Figure 4.37 Graph of the ellipse defined by x2+3y2+4y−4=0.x2+3y2+4y−4=0.

This equation implicitly defines yy as a function of x.x. As such, we can find the derivative dy/dxdy/dx using the method of implicit differentiation:

ddx(x2+3y2+4y−4)2x+6ydydx+4dydx(6y+4)dydxdydx====ddx(0)0−2x−x3y+2.ddx(x2+3y2+4y−4)=ddx(0)2x+6ydydx+4dydx=0(6y+4)dydx=−2xdydx=−x3y+2.

We can also define a function z=f(x,y)z=f(x,y) by using the left-hand side of the equation defining the ellipse. Then f(x,y)=x2+3y2+4y−4.f(x,y)=x2+3y2+4y−4. The ellipse x2+3y2+4y−4=0x2+3y2+4y−4=0 can then be described by the equation f(x,y)=0.f(x,y)=0. Using this function and the following theorem gives us an alternative approach to calculating dy/dx.dy/dx.

THEOREM 4.11

Implicit Differentiation of a Function of Two or More Variables

Suppose the function z=f(x,y)z=f(x,y) defines yy implicitly as a function y=g(x)y=g(x) of xx via the equation f(x,y)=0.f(x,y)=0. Then

dydx=−∂f/∂x∂f/∂ydydx=−∂f/∂x∂f/∂y

(4.34)

provided fy(x,y)≠0.fy(x,y)≠0.

If the equation f(x,y,z)=0f(x,y,z)=0 defines zz implicitly as a differentiable function of xandy,xandy, then

∂z∂x=−∂f/∂x∂f/∂zand∂z∂y=−∂f/∂y∂f/∂z∂z∂x=−∂f/∂x∂f/∂zand∂z∂y=−∂f/∂y∂f/∂z

(4.35)

as long as fz(x,y,z)≠0.fz(x,y,z)≠0.

Equation 4.34 is a direct consequence of Equation 4.31. In particular, if we assume that yy is defined implicitly as a function of xx via the equation f(x,y)=0,f(x,y)=0, we can apply the chain rule to find dy/dx:dy/dx:

ddxf(x,y)∂f∂x·dxdx+∂f∂y·dydx∂f∂x+∂f∂y·dydx===ddx(0)00.ddxf(x,y)=ddx(0)∂f∂x·dxdx+∂f∂y·dydx=0∂f∂x+∂f∂y·dydx=0.

Solving this equation for dy/dxdy/dx gives Equation 4.34. Equation 4.35 can be derived in a similar fashion.

Let's now return to the problem that we started before the previous theorem. Using Implicit Differentiation of a Function of Two or More Variables and the function f(x,y)=x2+3y2+4y−4,f(x,y)=x2+3y2+4y−4, we obtain

∂f∂x=2x∂f∂y=6y+4.∂f∂x=2x∂f∂y=6y+4.

Then Equation 4.34 gives

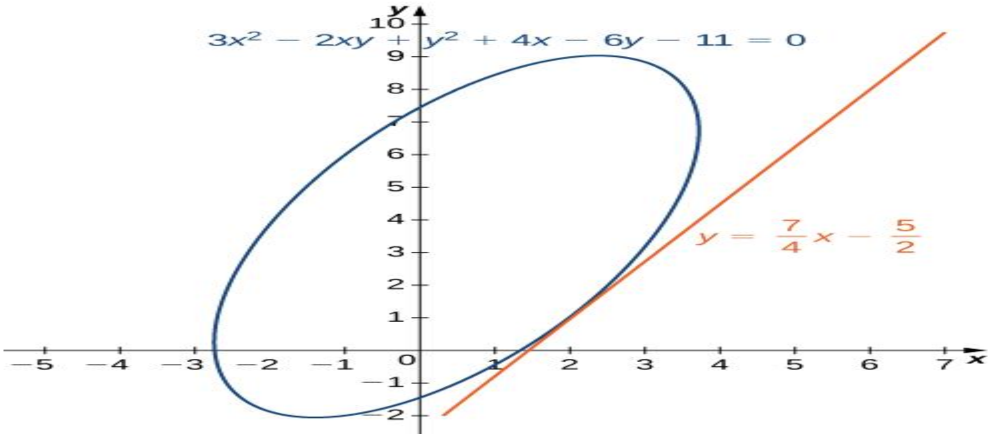
dydx=−∂f/∂x∂f/∂y=−2x6y+4=−x3y+2,dydx=−∂f/∂x∂f/∂y=−2x6y+4=−x3y+2,

which is the same result obtained by the earlier use of implicit differentiation.

EXAMPLE 4.30

Implicit Differentiation by Partial Derivatives

a. Calculate dy/dxdy/dx if yy is defined implicitly as a function of xx via the equation 3x2−2xy+y2+4x−6y−11=0.3x2−2xy+y2+4x−6y−11=0. What is the equation of the tangent line to the graph of this curve at point (2,1)?(2,1)?
b. Calculate ∂z/∂x∂z/∂x and ∂z/∂y,∂z/∂y, given x2ey−yzex=0.x2ey−yzex=0.



a.

CHECKPOINT 4.27

Find dy/dxdy/dx if yy is defined implicitly as a function of xx by the equation x2+xy−y2+7x−3y−26=0.x2+xy−y2+7x−3y−26=0. What is the equation of the tangent line to the graph of this curve at point (3,−2)?(3,−2)?

Section 4.5 Exercises

For the following exercises, use the information provided to solve the problem.

215.

Let w(x,y,z)=xycosz,w(x,y,z)=xycosz, where x=t,y=t2,x=t,y=t2, and z=arcsint.z=arcsint. Find d wdt.dwdt.

216.

Let w(t,v)=etvw(t,v)=etv where t=r+st=r+s and v=rs.v=rs. Find ∂w∂r∂w∂r and ∂w∂s.∂w∂s.

217.

If w=5x2+2y2,x=−3s+t,w=5x2+2y2,x=−3s+t, and y=s−4t,y=s−4t, find ∂w∂s∂w∂s and ∂w∂t.∂w∂t
.

218.

If w=xy2,x=5cos(2t),w=xy2,x=5cos(2t), and y=5sin(2t),y=5sin(2t), find dwdt.dwdt.

219.

If f(x,y)=xy,x=rcosθ,f(x,y)=xy,x=rcosθ, and y=rsinθ,y=rsinθ, find ∂f∂r∂f∂r and express the answer in terms of rr and θ.θ.

220.

Suppose f(x,y)=x+y,f(x,y)=x+y, where x=rcosθx=rcosθ and y=rsinθ.y=rsinθ. Find ∂f∂θ.∂f∂θ.

For the following exercises, find dfdtdfdt using the chain rule and direct substitution.

221.

f(x,y)=x2+y2,f(x,y)=x2+y2, x=t,y=t2x=t,y=t2

222.

f(x,y)=x2+y2−−−−−−√,y=t2,x=tf(x,y)=x2+y2,y=t2,x=t

223.

f(x,y)=xy,x=1−t√,y=1+t√f(x,y)=xy,x=1−t,y=1+t

224.

f(x,y)=xy,x=et,y=2etf(x,y)=xy,x=et,y=2et

225.

f(x,y)=ln(x+y),f(x,y)=ln(x+y), x=et,y=etx=et,y=et

226.

f(x,y)=x4,f(x,y)=x4, x=t,y=tx=t,y=t

227.

Let w(x,y,z)=x2+y2+z2,w(x,y,z)=x2+y2+z2, x=cost,y=sint,x=cost,y=sint, and z=et.z=et. Express ww as a function of tt and find dwdtdwdt directly. Then, find dwdtdwdt using the chain rule.

228.

Let z=x2y,z=x2y, where x=t2x=t2 and y=t3.y=t3. Find dzdt.dzdt.

229.

Let u=exsiny,u=exsiny, where x=−ln2tx=-
ln2t and y=πt.y=πt. Find dudtdudt when x=ln2x=ln2 and y=π4.y=π4.

For the following exercises, find dydxdydx using partial derivatives.

230.

sin(6x)+tan(8y)+5=0sin(6x)+tan(8y)+5=0

231.

x3+y2x−3=0x3+y2x−3=0

232.
sin(x+y)+cos(x−y)=4sin(x+y)+cos(x−y)=4
233.
x2−2xy+y4=4x2−2xy+y4=4
234.
xey+yex−2x2y=0xey+yex−2x2y=0
235.
x2/3+y2/3=a2/3x2/3+y2/3=a2/3
236.
xcos(xy)+ycosx=2xcos(xy)+ycosx=2
237.
exy+yey=1exy+yey=1
238.
x2y3+cosy=0x2y3+cosy=0
239.
Find dzdtdzdt using the chain rule where z=3x2y3,x=t4,z=3x2y3,x=t4, and y=t2.y=t2.
240.
Let z=3cosx−sin(xy),x=1t,z=3cosx−sin(xy),x=1t, and y=3t.y=3t. Find dzdt.dzdt.
241.
Let z=e1−xy,x=t1/3,z=e1−xy,x=t1/3, and y=t3.y=t3. Find dzdt.dzdt.
242.
Find dzdtdzdt by the chain rule where z=cosh2(xy),x=12t,z=cosh2(xy),x=12t, and y=et.y=et.
243.
Let z=xy,x=2cosu,z=xy,x=2cosu, and y=3sinv.y=3sinv. Find ∂z∂u∂z∂u and ∂z∂v.∂z∂v.
244.
Let z=ex2y,z=ex2y, where x=uv−−√x=uv and y=1v.y=1v. Find ∂z∂u∂z∂u and ∂z∂v.∂z∂v.
245.
If z=xyex/y,z=xyex/y, x=rcosθ,x=rcosθ, and y=rsinθ,y=rsinθ, find ∂z∂r∂z∂r and ∂z∂θ∂z∂θ when r=2r=2 and θ=π6.θ=π6.
246.
Find ∂w∂s∂w∂s if w=4x+y2+z3,x=ers2,y=ln(r+st),w=4x+y2+z3,x=ers2,y=ln(r+st), and z=rst2.z=rst2.
247.
If w=sin(xyz),x=1−3t,y=e1−t,w=sin(xyz),x=1−3t,y=e1−t, and z=4t,z=4t, find ∂w∂t.∂w∂t.

For the following exercises, use this information: A function f(x,y)f(x,y) is said to be homogeneous of degree nn if f(tx,ty)=tnf(x,y).f(tx,ty)=tnf(x,y). For all homogeneous functions of degree n,n, the following equation is true: x∂f∂x+y∂f∂y=nf(x,y).x∂f∂x+y∂f∂y=nf(x,y). Show that the given function is homogeneous and verify that x∂f∂x+y∂f∂y=nf(x,y).x∂f∂x+y∂f∂y=nf(x,y).

248.
f(x,y)=3x2+y2f(x,y)=3x2+y2
249.
f(x,y)=x2+y2−−−−−−√f(x,y)=x2+y2
250.
f(x,y)=x2y−2y3f(x,y)=x2y−2y3
251.

The volume of a right circular cylinder is given by V(x,y)=πx2y,V(x,y)=πx2y, where xx is the radius of the cylinder and *y* is the cylinder height. Suppose xx and yy are functions of tt given by x=12tx=12t and y=13ty=13t so that xandyxandy are both increasing with time. How fast is the volume increasing when x=2x=2 and y=43?y=43?

252.

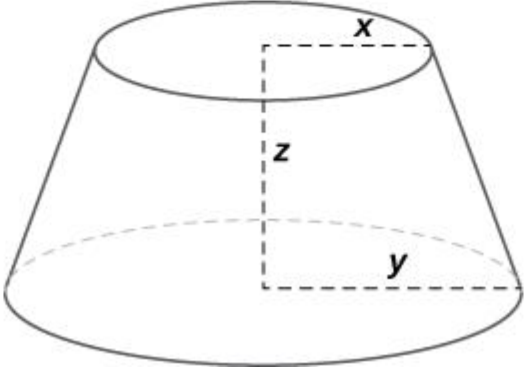
The pressure PP of a gas is related to the volume and temperature by the formula PV=kT,PV=kT, where temperature is expressed in kelvins. Express the pressure of the gas as a function of both VV and T.T. Find dPdtdPdt when k=1,k=1, dVdt=2dVdt=2 cm^3/min, dTdt=12dTdt=12 K/min, V=20V=20 cm^3, and T=20°F.T=20°F.

253.

The radius of a right circular cone is increasing at 33 cm/min whereas the height of the cone is decreasing at 22 cm/min. Find the rate of change of the volume of the cone when the radius is 1313 cm and the height is 1818 cm.

254.

The volume of a frustum of a cone is given by the formula V=13πz(x2+y2+xy),V=13πz(x2+y2+xy), where xx is the radius of the smaller circle, yy is the radius of the larger circle, and zz is the height of the frustum (see figure). Find the rate of change of the volume of this frustum when x=10in.,y=12in.,andz=18in.x=10in.,y=12in.,andz=18in. if dzdt=−5,dxdt=1,dydt=1dzdt=-5,dxdt=1,dydt=1 (all in/min).



255.

A closed box is in the shape of a rectangular solid with dimensions x,y,andz.x,y,andz. (Dimensions are in inches.) Suppose each dimension is changing at the rate of 0.50.5 in./min. Find the rate of change of the total surface area of the box when x=2in.,y=3in.,andz=1in.x=2in.,y=3in.,andz=1in.

256.

The total resistance in a circuit that has three individual resistances represented by x,y,x,y, and zz is given by the formula R(x,y,z)=xyzyz+xz+xy.R(x,y,z)=xyzyz+xz+xy. Suppose at a given time the xx resistance is 100Ω,100Ω, the *y* resistance is 200Ω,200Ω, and the zz resistance is 300Ω.300Ω. Also, suppose the xx resistance is changing at a rate of 2Ω/min,2Ω/min, the yy resistance is changing at the rate of 1Ω/min,1Ω/min, and

the zz resistance has no change. Find the rate of change of the total resistance in this circuit at this time.
257.

The temperature TT at a point (x,y)(x,y) is T(x,y)T(x,y) and is measured using the Celsius scale. A fly crawls so that its position after tt seconds is given by x=1+t———√x=1+t and y=2+13t,y=2+13t, where xandyxandy are measured in centimeters. The temperature function satisfies Tx(2,3)=4Tx(2,3)=4 and Ty(2,3)=3.Ty(2,3)=3. How fast is the temperature increasing on the fly's path after 33 sec?
258.

The xandyxandy components of a fluid moving in two dimensions are given by the following functions: u(x,y)=2yu(x,y)=2y and v(x,y)=−2x;v(x,y)=−2x; x≥0;y≥0.x≥0;y≥0. The speed of the fluid at the point (x,y)(x,y) is s(x,y)=u(x,y)2+v(x,y)2——————————√.s(x,y)=u(x,y)2+v(x,y)2. Find ∂s∂x∂s∂x and ∂s∂y∂s∂y using the chain rule.
259.

Let u=u(x,y,z),u=u(x,y,z), where x=x(w,t),y=y(w,t),z=z(w,t),w=w(r,s),andt=t(r,s).x=x(w,t),y=y(w,t),z=z(w,t),w=w(r,s),andt=t(r,s). Use a tree diagram and the chain rule to find an expression for ∂u∂r.

QUESTION#04: (06) Let $w = \ln(e\ r + e\ s + e\ t + e\ u)$. $Show\ that\ wrstu = -6\ e\ r+s+t+u-4$

Let w(x,y,z)=xycosz,w(x,y,z)=xycosz, where x=t,y=t2,x=t,y=t2, and z=arcsint.z=arcsint. Find dwdt.dwdt.
216.

Let w(t,v)=etvw(t,v)=etv where t=r+st=r+s and v=rs.v=rs. Find ∂w∂r∂w∂r and ∂w∂s.∂w∂s.
217.

If w=5x2+2y2,x=−3s+t,w=5x2+2y2,x=−3s+t, and y=s−4t,y=s−4t, find ∂w∂s∂w∂s and ∂w∂t.∂w∂t.
218.

If w=xy2,x=5cos(2t),w=xy2,x=5cos(2t), and y=5sin(2t),y=5sin(2t), find dwdt.dwdt.
219.

If f(x,y)=xy,x=rcosθ,f(x,y)=xy,x=rcosθ, and y=rsinθ,y=rsinθ, find ∂f∂r∂f∂r and express the answer in terms of rr and θ.θ.
220.

Suppose f(x,y)=x+y,f(x,y)=x+y, where x=rcosθx=rcosθ and y=rsinθ.y=rsinθ. Find ∂f∂θ.∂f∂θ.

For the following exercises, find dfdtdfdt using the chain rule and direct substitution.

221.

f(x,y)=x2+y2,f(x,y)=x2+y2, x=t,y=t2x=t,y=t2

222.

f(x,y)=x2+y2−−−−−−√,y=t2,x=tf(x,y)=x2+y2,y=t2,x=t

223.

f(x,y)=xy,x=1−t√,y=1+t√f(x,y)=xy,x=1−t,y=1+t

224.

f(x,y)=xy,x=et,y=2etf(x,y)=xy,x=et,y=2et

225.

f(x,y)=ln(x+y),f(x,y)=ln(x+y), x=et,y=etx=et,y=et

226.

f(x,y)=x4,f(x,y)=x4, x=t,y=tx=t,y=t

227.

Let w(x,y,z)=x2+y2+z2,w(x,y,z)=x2+y2+z2, x=cost,y=sint,x=cost,y=sint, and z=et.z=et. Express ww as a function of tt and find dwdtdwdt directly. Then, find dwdtdwdt using the chain rule.

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For the following exercises, find dydxdydx using partial derivatives.

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sin(6x)+tan(8y)+5=0sin(6x)+tan(8y)+5=0

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sin(x+y)+cos(x−y)=4sin(x+y)+cos(x−y)=4

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235.

x2/3+y2/3=a2/3x2/3+y2/3=a2/3
236.

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237.

exy+yey=1exy+yey=1
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Let z=3cosx−sin(xy),x=1t,z=3cosx−sin(xy),x=1t, and y=3t.y=3t. Find dzdt.dzdt.
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Let z=e1−xy,x=t1/3,z=e1−xy,x=t1/3, and y=t3.y=t3. Find dzdt.dzdt.
242.

Find dzdtdzdt by the chain rule where z=cosh2(xy),x=12t,z=cosh2(xy),x=12t, and y=et.y=et.
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Let z=xy,x=2cosu,z=xy,x=2cosu, and y=3sinv.y=3sinv. Find ∂z∂u∂z∂u and ∂z∂v.∂z∂v.
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Let z=ex2y,z=ex2y, where x=uv−−√x=uv and y=1v.y=1v. Find ∂z∂u∂z∂u and ∂z∂v.∂z∂v.
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Find ∂w∂s∂w∂s if w=4x+y2+z3,x=ers2,y=ln(r+st),w=4x+y2+z3,x=ers2,y=ln(r+st), and z=rst2.z=rst2.
247.

If w=sin(xyz),x=1−3t,y=e1−t,w=sin(xyz),x=1−3t,y=e1−t, and z=4t,z=4t, find ∂w∂t.∂w∂t.

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f(x,y)=3x2+y2f(x,y)=3x2+y2
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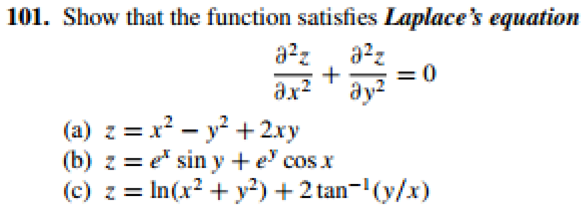
QUESTION#05: (06) Show that function in satisfies a LAPLACE equation: z = ln(x 2 + y 2) + 2 tan−1 (y x)

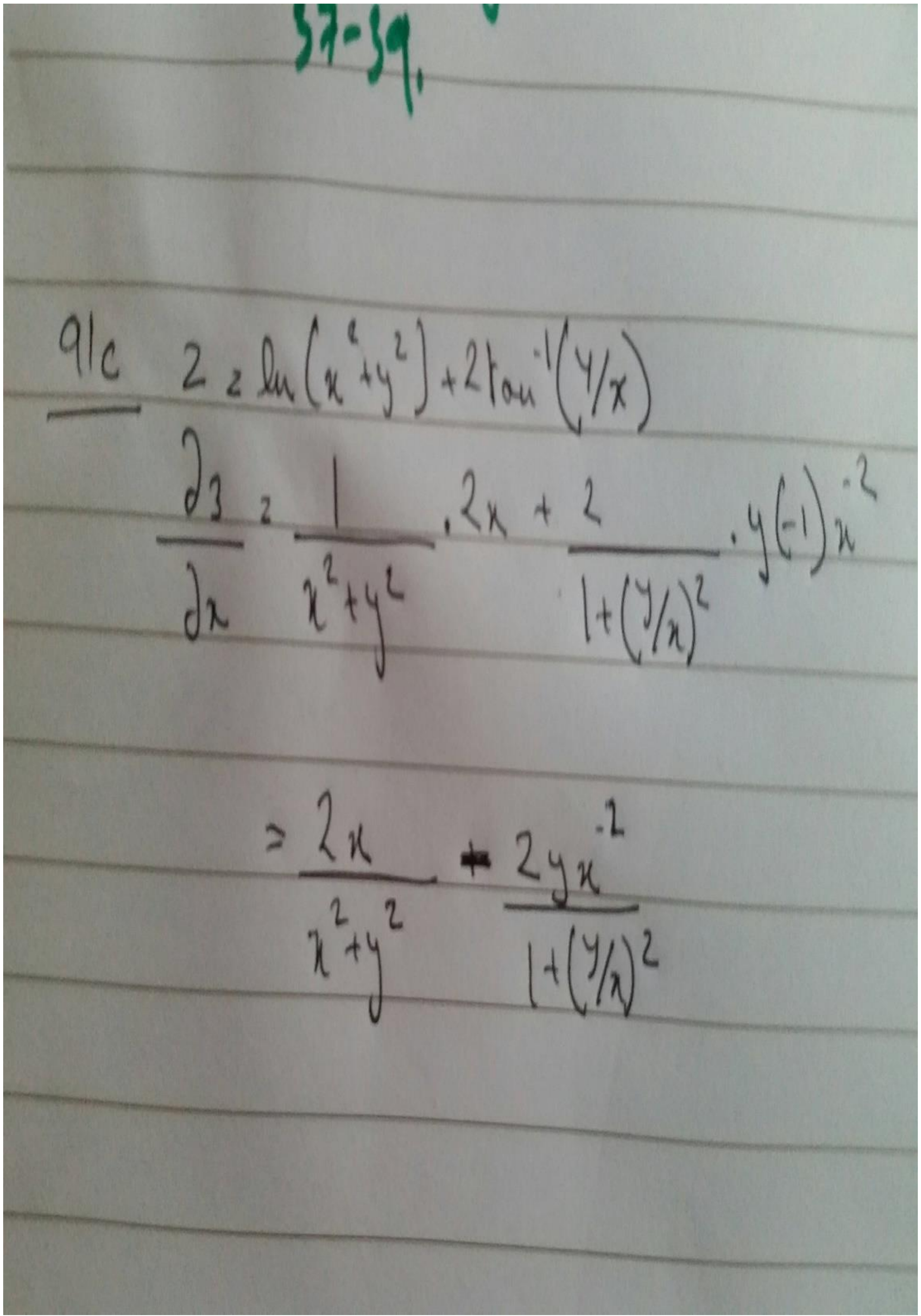


101. Show that the function satisfies ***Laplace's equation***

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$$

(a) $z = x^2 - y^2 + 2xy$
(b) $z = e^x \sin y + e^y \cos x$
(c) $z = \ln(x^2 + y^2) + 2\tan^{-1}(y/x)$

I can get to the first partial derivative of ∂z/∂x∂z/∂x of course, but after that I get this, which I don't know how to differentiate further... I am talking about c part of the question. Attaching my working...



37-39.

9|c $z = \ln(x^2+y^2) + 2\tan^{-1}(y/x)$

$$\frac{\partial z}{\partial x} = \frac{1}{x^2+y^2} \cdot 2x + \frac{2}{1+(y/x)^2} \cdot y(-1)x^{-2}$$

$$= \frac{2x}{x^2+y^2} + \frac{2yx^{-2}}{1+(y/x)^2}$$

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- multivariable-calculus

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edited Jun 4, 2016 at 12:22
asked Jun 4, 2016 at 12:07

studious
34522 gold badges55 silver badges1313 bronze badges

- 2
  Will it not be easier using polar coordinate form of Laplace equation 2(logr+θ)?2(logr+θ)?
  – Narasimham
  Jun 4, 2016 at 12:23
- possibly, but this is not covered in class, would rather go through a 'basic' elementary route.
  – studious
  Jun 4, 2016 at 12:24
- 1
  @studious I left an answer doing it the "brute force" way. Tell me in case anything is not clear.
  – qmd
  Jun 4, 2016 at 13:06

Add a comment
1 Answer
Sorted by:
Differentiating once:
∂z∂x(ln(x2+y2)+2tan−1(yx))=2xx2+y2+−2yx2x2+y2x2=2xx2+y2−2yx2+y2=2x−2yx2+y2∂z∂x(ln(x2+y2)+2tan−1(yx))=2xx2+y2+−2yx2x2+y2x2=2xx2+y2−2yx2+y2=2x−2yx2+y2
Differentiating again:
∂z∂x(∂z∂x)=∂∂x(2x−2yx2+y2)=2(x2+y2)−2x(2x−2y)(x2+y2)2=−2x2+2y2+2xy(x2+y2)2∂z∂x(∂z∂x)=∂∂x(2x−2yx2+y2)=2(x2+y2)−2x(2x−2y)(x2+y2)2=−2x2+2y2+2xy(x2+y2)2

Now let us show ∂2z∂y2∂2z∂y2:
∂∂y(ln(x2+y2)+2tan−1(yx))=2yx2+y2+1x1+(yx)2=2y+xx2+y2∂∂y(ln(x2+y2)+2tan−1(yx))=2yx2+y2+1x1+(yx)2=2y+xx2+y2
∂∂y(2y+xx2+y2)=2(x2+y2)−(2y)(2y+x)(x+y)2=2x2−2y2−2xy(x2+y2)2∂∂y(2y+xx2+y2)=2(x2+y2)−(2y)(2y+x)(x+y)2=2x2−2y2−2xy(x2+y2)2

Therefore,
∂2z∂x2+∂2z∂y2=−2x2+2y2+2xy(x2+y2)2+2x2−2y2−2xy(x2+y2)2=0✓
.