Mechanics of Materials

Chapter 3

Torsion

3.1 *Introduction*

- In many engineering applications, members are required to carry **torsional loads**.
- Consider the torsion of circular shafts. Because a circular cross section is an efficient shape for resisting torsional loads. Circular shafts are commonly used to transmit power in rotating machinery.
- Also discuss another important application—torsion of thin-walled tubes..

3.1 *Torsion of Circular Shafts*

a. Simplifying assumptions

- During the deformation, the cross sections are not distorted in any manner—they remain plane, and the radius r does not change. In addition, the length L of the shaft remains constant.

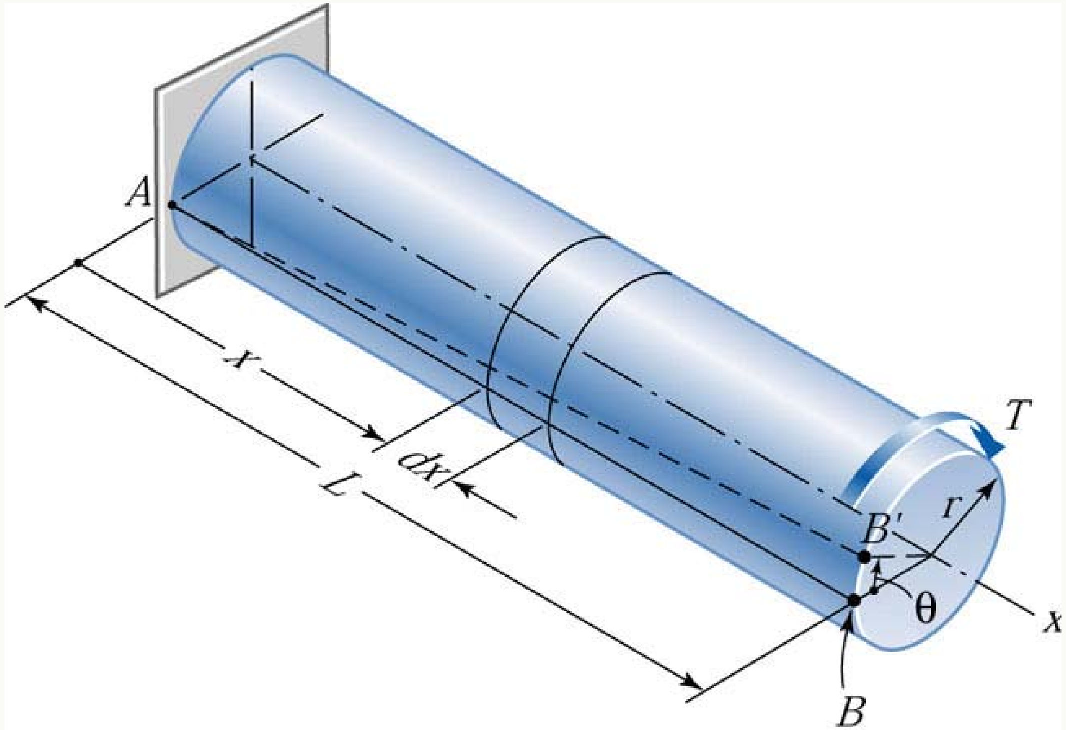
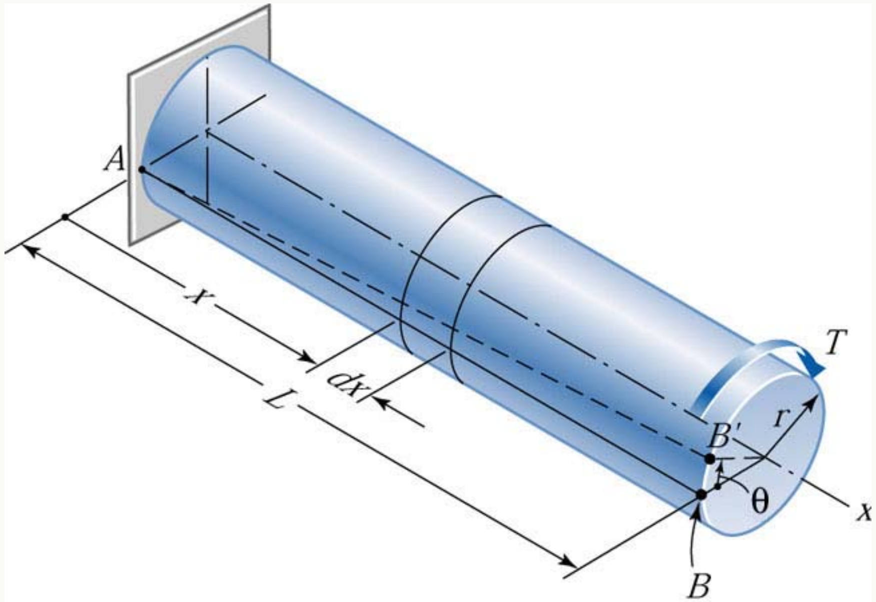


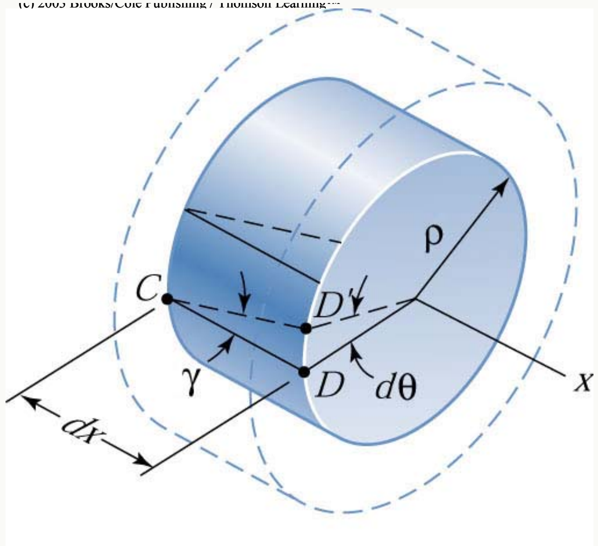
Figure 3.1 Deformation of a circular shaft caused by the torque *T*. The initially straight line *AB* deforms into a helix.

- Based on these observations, we make the following assumptions:
  - Circular cross sections remain plane (do not warp) and perpendicular to the axis of the shaft.
  - Cross sections do not deform (there is no strain in the plane of the cross section).
  - The distances between cross sections do not change (the axial normal strain is zero).
- *Each cross section rotates as a rigid entity about the axis of the shaft*. Although this conclusion is based on the observed deformation of a cylindrical shaft carrying a **constant** internal torque, we assume that the result remains valid even if the diameter of the shaft or the internal torque **varies** along the length of the shaft.

b. *Compatibility*

- Because the cross sections are separated by an infinitesimal distance, the difference in their rotations, denoted by the angle $d\theta$, is also infinitesimal.
- As the cross sections undergo the relative rotation $d\theta$, *CD* deforms into the helix *CD*. By observing the distortion of the shaded element, we recognize that the helix angle γ is the *shear strain* of the element.





(a)

From the geometry of Fig.3.2(a), we obtain DD´ = $\rho\, d\theta = \gamma\, dx$, from which the shear strain γ is

$$\gamma = \frac{d\theta}{dx}\rho \qquad (3.1)$$

The quantity $d\theta/dx$ is the *angle of twist per unit length,* where θ is expressed in radians. The corresponding shear stress, illustrated in Fig. 3.2 (b), is determined from Hooke´s law:

$$\tau = G\gamma = G\frac{d\theta}{dx}\rho \qquad (3.2)$$

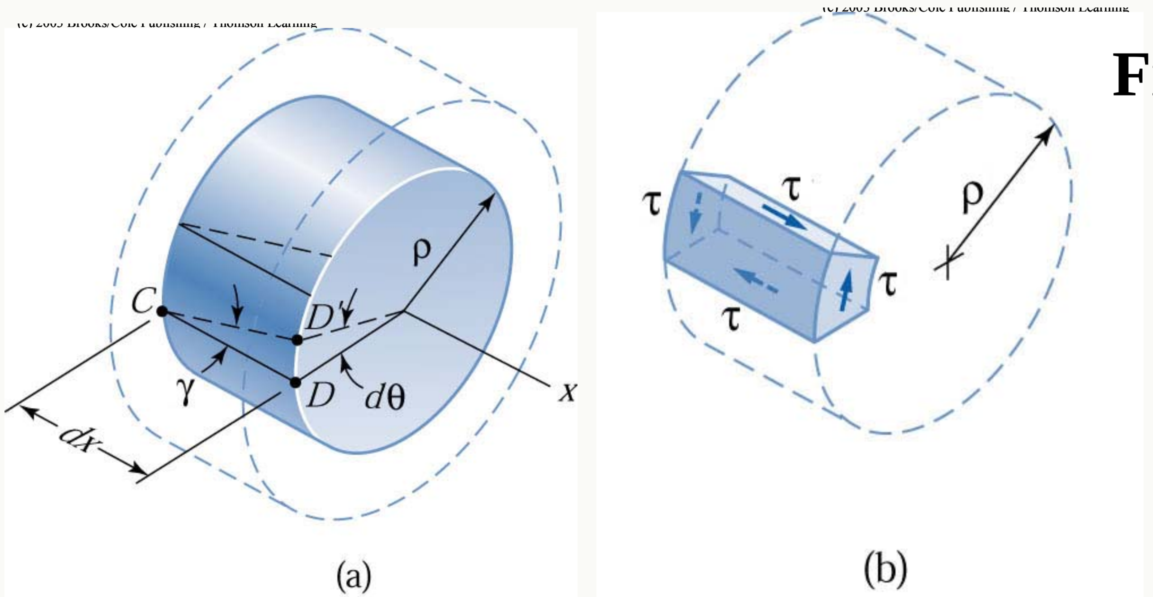


Figure 3.2 (a) Shear strain of a material element caused by twisting of the shaft; (b) the corresponding shear stress.

- *the shear stress varies linearly with the radial distance ρ from the axial of the shaft.* $\tau = G\gamma = G\frac{d\theta}{dx}\rho$
- The variation of the shear stress acting on the cross section is illustrated in Fig. 3.3. The maximum shear stress, denoted by τ_{max}, occurs at the surface of the shaft.
- Note that the above derivations assume neither a constant internal torque nor a constant cross section along the length of the shaft.

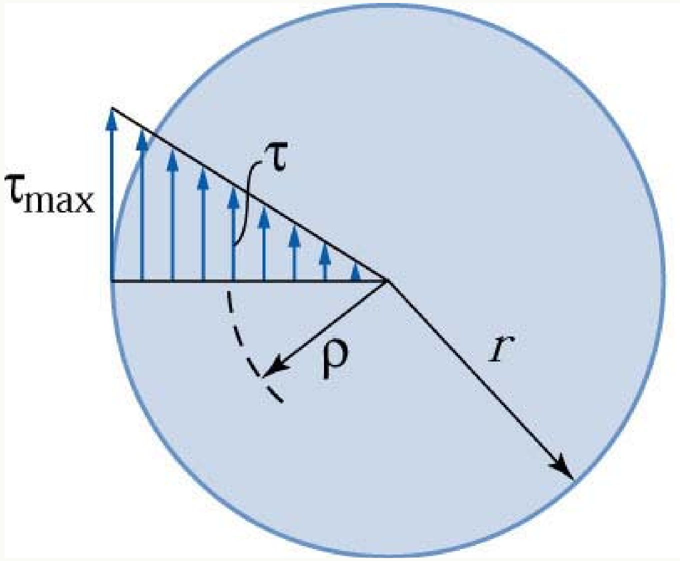


Figure 3.3 Distribution of shear stress along the radius of a circular shaft.

c. *Equilibrium*

- Fig. 3.4 shows a cross section of the shaft containing a differential element of area *dA* loaded at the radial distance ρ from the axis of the shaft.

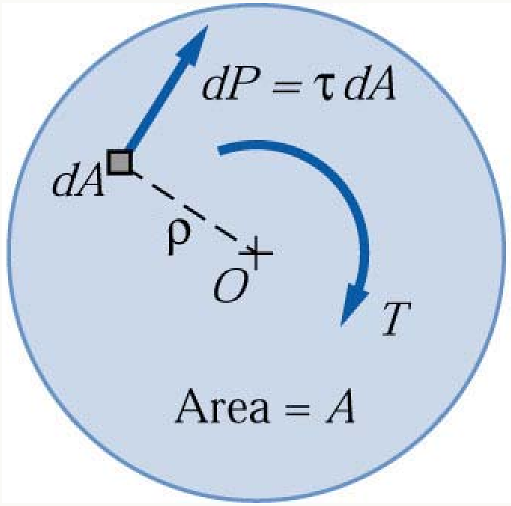


Figure 3.4 Calculating the Resultant of the shear stress acting on the cross section. Resultant is a couple equal to the internal torque *T*.

- The shear force acting on this area is $dP = \tau\, dA = G\,(d\theta/dx)\,\rho\, dA$, directed perpendicular to the radius. Hence, the moment (torque) of dP about the center O is $\rho\, dP = G\,(d\theta/dx)\,\rho^2 dA$. Summing the contributions and equating the result to the internal torque yields $\int_A \rho dP = T$, or

$$G\frac{d\theta}{dx}\int_A \rho^2 dA = T$$

Recognizing that is the polar moment of inertia of the cross-sectional area, we can write this equation as $G\,(d\theta/dx)\,J = T$, or

$$\frac{d\theta}{dx} = \frac{T}{GJ} \qquad (3.3)$$

The rotation of the cross section at the free end of the shaft, called **the angle of twist** θ , is obtained by integration:

$$\theta = \int_o^L d\theta = \int_o^L \frac{T}{GJ}dx \qquad (3.4a)$$

As in the case of a **prismatic bar** carrying a constant torque, then reduces **the torque-twist relationship**

$$\theta = \frac{TL}{GJ} \qquad (3.4b)$$

Note the similarity between Eqs. (3.4) and the corresponding formulas for axial deformation: $\delta = \int_o^L (P/EA)dx$ and $\delta = PL/(EA)$

Notes on the Computation of angle of Twist

- 1.In the U.S. Customary system, the consistent units are G [psi], T [$lb \cdot$ in] , and L [in.], and J [in.^{4}]; in the SI system, the consistent units are G [Pa], T [N · m], L [m], and J [m^4].
- 2.The unit of θ in Eqs. (3.4) is radians, regardless of which system of unit is used in the computation.
- 3.Represent torques as vectors using the right-hand rule, as illustrated in Fig. 3.5. The same sign convention applies to the angle of twist θ .

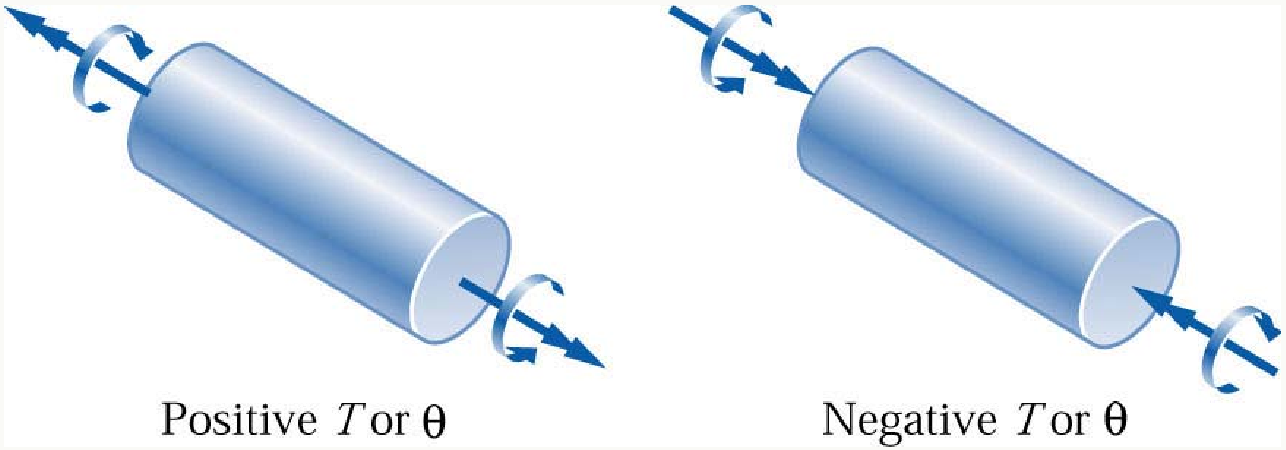


Figure 3.5 Sign Conventions for Torque *T* and angle of twist *T*.

d. Torsion formulas

- $G\,(d\theta/dx) = T/J$, which substitution into Eq. (3.2), $\tau = G\gamma = G\dfrac{d\theta}{dx}\rho$ gives the shear stress τ acting at the distance ρ from the center of the shaft, ***Torsion formulas*** :

$$\tau = \frac{T\rho}{J} \qquad (3.5a)$$

The **maximum** shear stress τ_{max} is found by replacing ρ by the radius r of the shaft:

$$\tau_{max} = \frac{Tr}{J} \qquad (3.5b)$$

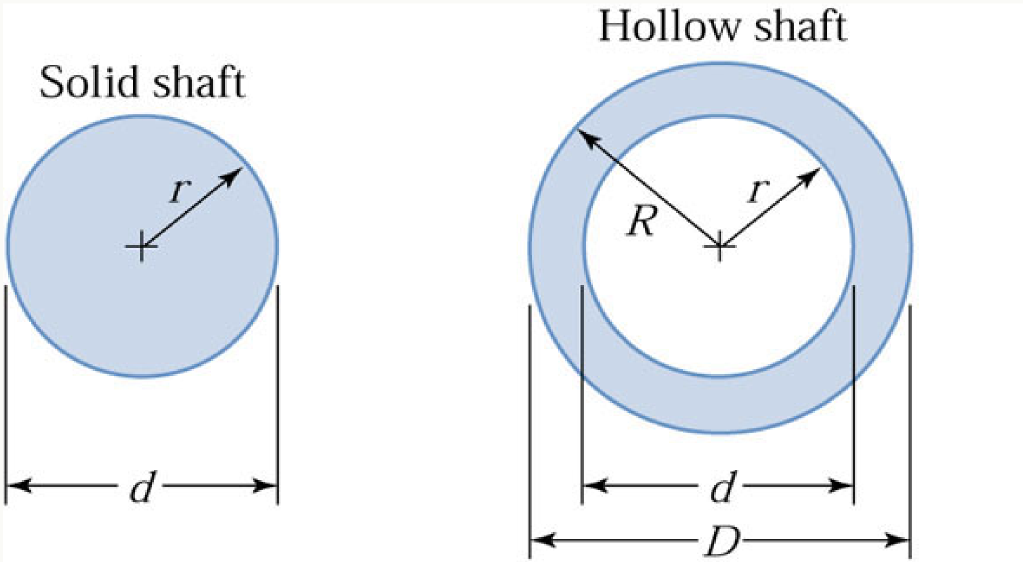
- Because Hook´s law was used in the derivation of Eqs. (3.2)-(3.5), these formulas are valid if the shear stresses do not exceed the proportional limit of the material shear. Furthermore, these formulas are applicable only to **circular shafts**, either solid or hollow.

- The expressions for the polar moments of circular areas are :

Solid shaft : $$\tau_{max} = \frac{2T}{\pi r^3} = \frac{16T}{\pi d^3} \tag{3.5c}$$

Hollow shaft : $$\tau_{max} = \frac{2TR}{\pi\left(R^4 - r^4\right)} = \frac{16TD}{\pi\left(D^4 - d^4\right)} \tag{3.5d}$$

Equations (3.5c) and (3.5d) are called the ***torsion formulas***.



$$J = \frac{\pi r^4}{2} = \frac{\pi d^4}{32} \qquad J = \frac{\pi}{2}(R^4 - r^4) = \frac{\pi}{32}(D^4 - d^4)$$

Figure 3.6 Polar moments of inertia of circular areas.

e. *Power transmission*

- Shafts are used to transmit power. The power ζ transmitted by a torque T rotating at the angular speed ω is given by $\zeta = T\omega$, where ω is measured in radians per unit time.
- If the shaft is rotating with a frequency of f revolutions per unit time, then $\omega = 2\pi f$, which gives $\zeta = T(2\pi f)$. Therefore, the torque can be expressed as

$$T = \frac{\zeta}{2\pi f} \qquad (3.6a)$$

- In SI units, ζ in usually measured in watts (1.0 W=1.0 N・m/s) and f in hertz (1.0 Hz = 1.0 rev/s); Eq. (3.6a) then determines the torque T in N・m.
- In U.S. Customary units with ζ in *lb*・in./s and f in hertz, Eq.(3.6a) calculates the torque T in *lb*・in.

- Because power in U.S. Customary units is often expressed in horsepower (1.0 hp = 550 *l*b • ft/s = 396×10^3 *l*b • in./min), a convenient form of Eq.(3.6a) is

$$T(lb \cdot in) = \frac{\zeta(hp)}{2\pi f(rev/\min)} \times \frac{396 \times 10^3 (lb \cdot in./\min)}{1.0(hp)}$$

which simplifies to

$$T(lb \cdot in) = 63.0 \times 10^3 \frac{\zeta(hp)}{f(rev/\min)} \quad (3.6b)$$

f. *Statically indeterminate problems*

- Draw the required **free-body diagrams** and write the equations of **equilibrium**.
- Derive the **compatibility** equations from the restrictions imposed on the angles of twist.
- Use the **torque- twist relationships** in Eqs.(3.4) to express the angles of twist in the compatibility equations in terms of the torques.
- **Solve the equations** of equilibrium and compatibility for the torques.

Sample Problem 3.1

A solid steel shaft in a rolling mill transmits 20 kW of power at 2 Hz. Determine the smallest safe diameter of the shaft if the shear stress τ_w is not to exceed 40 MPa and the angle of twist θ is limited to 6° in a length of 3 m. Use G = 83 GPa.

Solution

Applying Eq. (3.6a) to determine the torque:

$$T = \frac{P}{2\pi f} = \frac{20 \times 10^3}{2\pi(2)} = 1591.5 N \cdot m$$

To satisfy the strength condition, we apply the torsion formula, Eq. (3.5c):

$$\tau_{max} = \frac{Tr}{J} \quad \tau_{max} = \frac{16T}{\pi d^3} \qquad 4 \times 10^6 = \frac{16(1591.5)}{\pi d^3}$$

Which yields $d = 58.7 \times 10^{-3}$ m = 58.7 mm.

Apply the torque-twist relationship, Eq. (3.4b), to determine the diameter necessary to satisfy the requirement of rigidity (remembering to convert θ from degrees to radians):

$$\theta = \frac{TL}{GJ} \qquad 6\left(\frac{\pi}{180}\right) = \frac{1591.5(3)}{(83\times10^{9})(\pi d^{4}/32)}$$

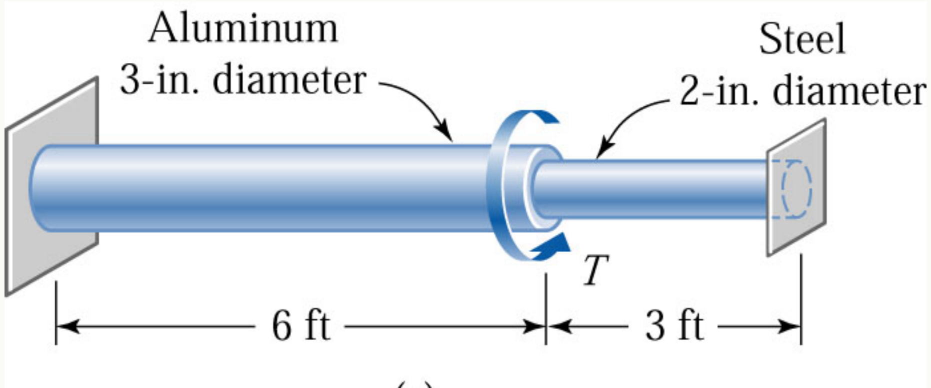
From which we obtain $d = 48.6\times10^{-3}$ m = 48.6 mm.

To satisfy both strength and rigidity requirements, we must choose the larger diameter-namely,

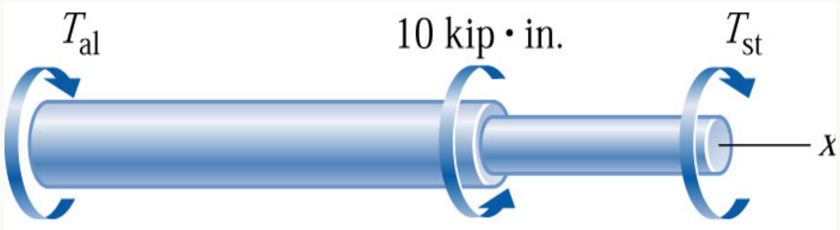
$d = 58.7$ mm. *Answer*

Sample problem 3.2

The shaft in Fig. (a) consists of a 3-in. -diameter aluminum segment that is rigidly joined to a 2-in. -diameter steel segment. The ends of the shaft are attached to rigid supports, Calculate the maximum shear stress developed in each segment when the torque $T = 10$ kip in. is applied. Use $G = 4\times10^6$ psi for aluminum and $G = 12\times10^6$ psi for steel.



(a)



(b) FBD

Solution

Equilibrium $\Sigma M_x = 0\,, \quad (10\times10^3) - T_{st} - T_{al} = 0 \qquad (a)$

This problem is statically indeterminate.

Compatibility the two segments must have the same angle of twist; that is, $\theta_{st} = \theta_{al}$ From Eq. (3.4b), this condition between.

$$\left(\frac{TL}{GJ}\right)_{st} = \left(\frac{TL}{GJ}\right)_{al} \qquad \frac{T_{st}(3\times 12)}{(12\times 10^6)\frac{\pi}{32}(2)^4} = \frac{T_{al}(6\times 12)}{(4\times 10^6)\frac{\pi}{32}(3)^4}$$

from which

$$T_{st} = 1.1852\ T_{al} \qquad \text{(b)}$$

Solving Eqs. (a) and (b), we obtain

$$T_{al} = 4576\ lb\cdot\ \text{in.} \qquad T_{st} = 5424\ lb\cdot\ \text{in.}$$

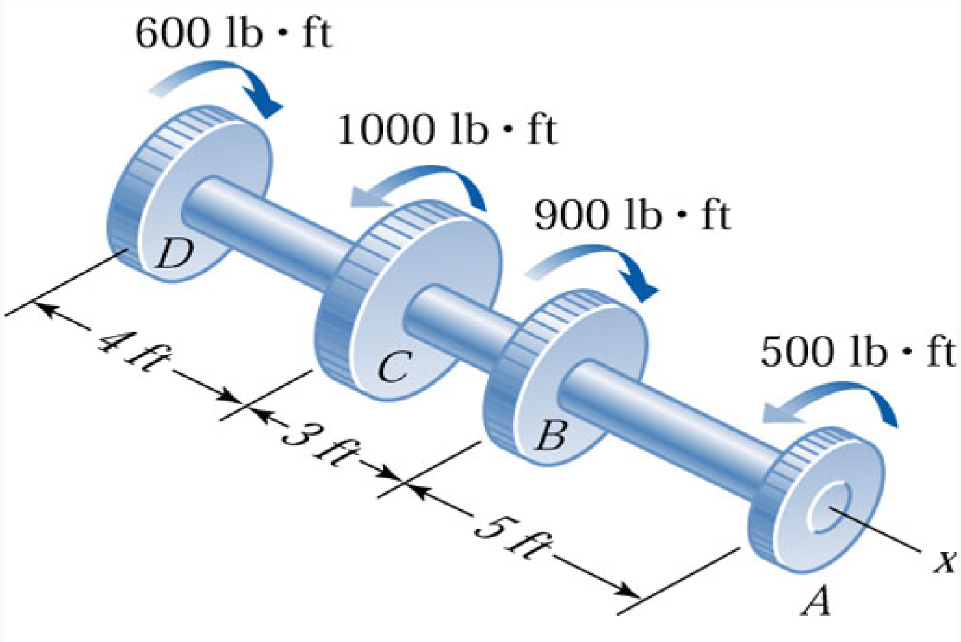
the maximum shear stresses are

$$(\tau_{max})_{al} = \left(\frac{16T}{\pi d^3}\right)_{al} = \frac{16(4576)}{\pi(3)^3} = 863\,psi \qquad \textit{Answer}$$

$$(\tau_{max})_{st} = \left(\frac{16T}{\pi d^3}\right)_{st} = \frac{16(5424)}{\pi(2)^3} = 3450\,psi \qquad \textit{Answer}$$

***Sample problem* 3.3**

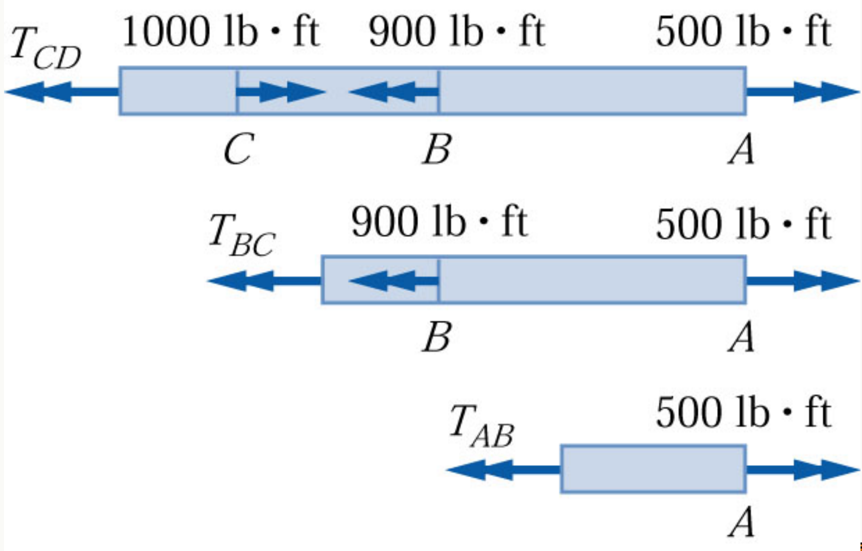
The four rigid gears, loaded as shown in Fig. (a), are attached to a 2-in.-diameter steel shaft. Compute the angle θ of rotation of gear A relative to gear D. Use $G = 12\times10^6$ psi for the shaft.



(a)

Solution

It is convenient to represent the torques as vectors (using the right-hand rule) on the FBDs in Fig. (b).



(b) FBDs

Solution

Assume that the internal torques T_{AB}, T_{BC}, and T_{CD} are positive according to the sign convention introduced earlier (positive torque vectors point away from the cross section). Applying the equilibrium condition $\sum M_x = 0$ to each FBD, we obtain

$$500 - 900 + 1000 - T_{CD} = 0$$

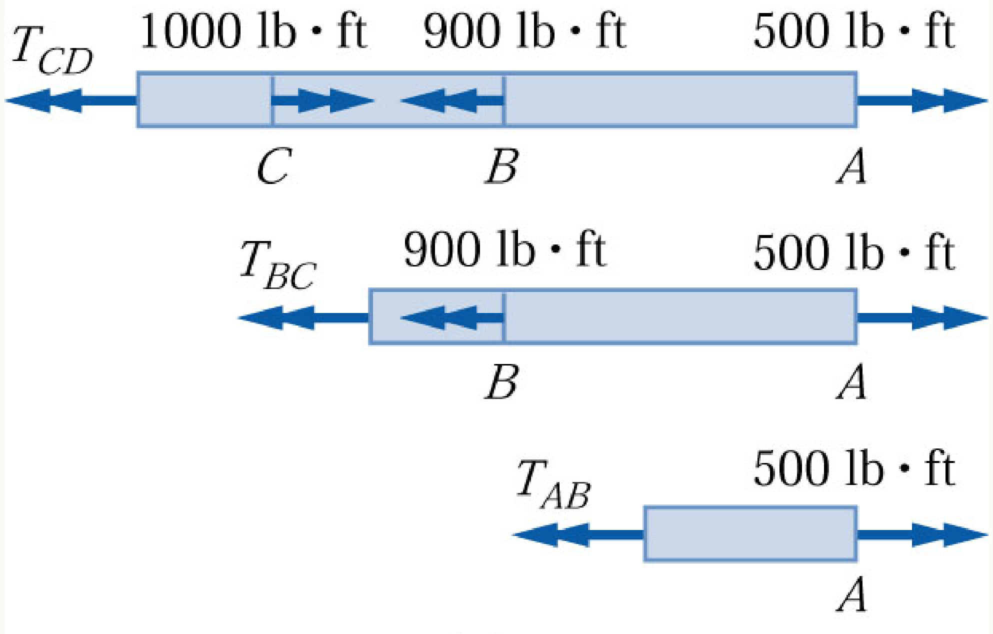
$$500 - 900 - T_{BC} = 0$$

$$500 - T_{AB} = 0$$

$$T_{AB} = 500\ lb \cdot \text{ft},$$

$$T_{BC} = -400\ lb \cdot \text{ft}$$

$$T_{CD} = 600\ lb \cdot \text{ft}$$



(b) FBDs

The minus sign indicates that the sense of T_{BC} is opposite to that shown on the FBD.*A* is gear *D* were fixed.

This rotation is obtained by summing the angles of twist of the three segments:

$$\theta_{A/D} = \theta_{A/B} + \theta_{B/C} + \theta_{C/D}$$

Using Eq.(3.4b), we obtain (converting the lengths to inches and torques to pound-inches)

$$\theta_{A/D} = \frac{T_{AB}L_{AB} + T_{BC}L_{BC} + T_{CD}L_{CD}}{GJ}$$

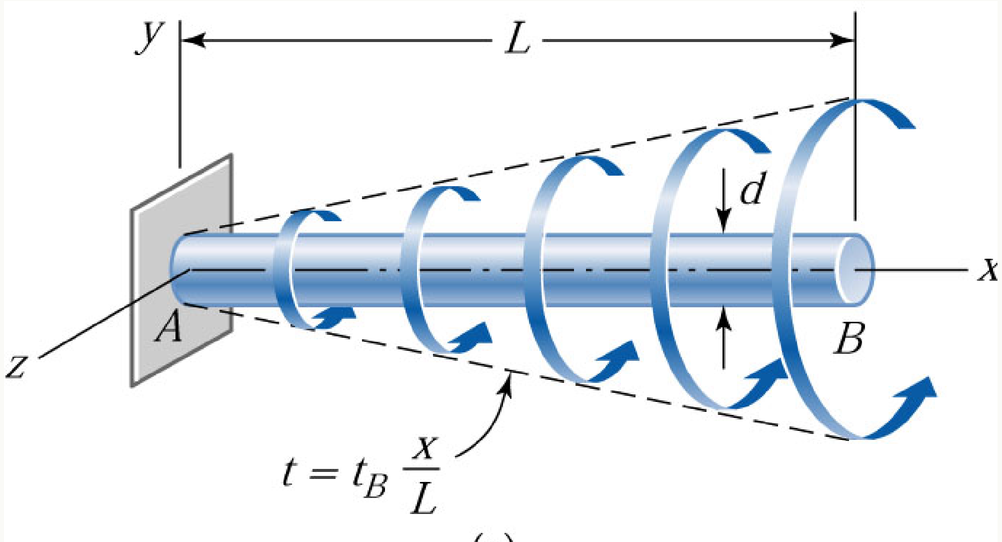
$$= \frac{(500\times12)(5\times12) - (400\times12)(3\times12) + (600\times12)(4\times12)}{\left[\pi(2)^4/32\right](12\times10)^6}$$

$$= 0.02827 \text{ rad} = 1.620^\circ \qquad \textit{Answer}$$

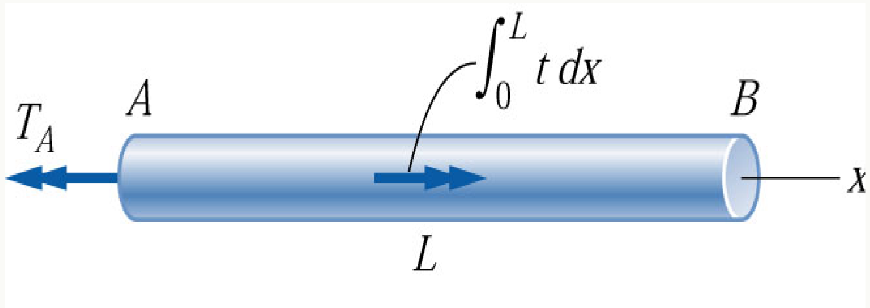
The positive result indicates that the rotation vector of *A* relative to *D* is in the positive *x*-direction: that is, θ_{AD} is directed counterclockwise when viewed from *A* toward *D*.

Sample Problem **3.4**

Figure (a) shows a steel shaft of length $L = 1.5$ m and diameter $d = 25$ mm that carries a distributed toque of intensity (torque per unit length) $t = t_B(x/L)$, where $t_B = 200$ N· m/m. Determine (1) the maximum shear stress in the shaft; and (2) the angle of twist. Use $G = 80$ GPa for steel.



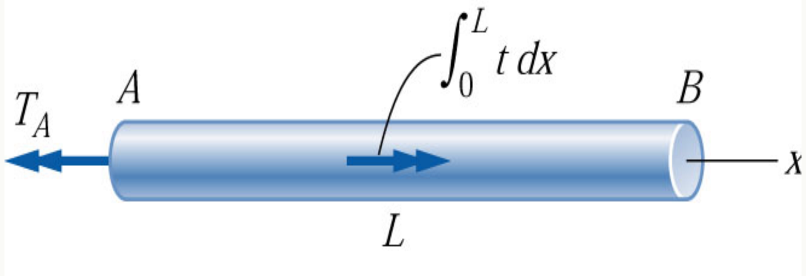
(a)



(b) FBD

Figure (a) and (b) FBD

Solution



(b) FBD

Part 1

Figure (b) shows the FBD of the shaft. The total torque applied to the shaft is $\int_0^t tdx$. The maximum torque in the shaft is T_A, which occurs at the fixed support. From the FBD we get

$$\sum M_X = 0 \qquad \int_0^L tdx - T_A = 0$$

$$T_A = \int_0^L tdx = \int_0^L t_B \frac{x}{L} dx = \frac{t_B L}{2} = \frac{1}{2}(200)(1.5) = 150 N \cdot m$$

From Eq. (3.5c), the maximum stress in the shaft is

$$\tau_{\max} = \frac{16T_A}{\pi d^3} = \frac{16(150)}{\pi(0.025)^3} = 48.9 \times 10^6\, Pa = 48.9 MPa \quad Answer$$

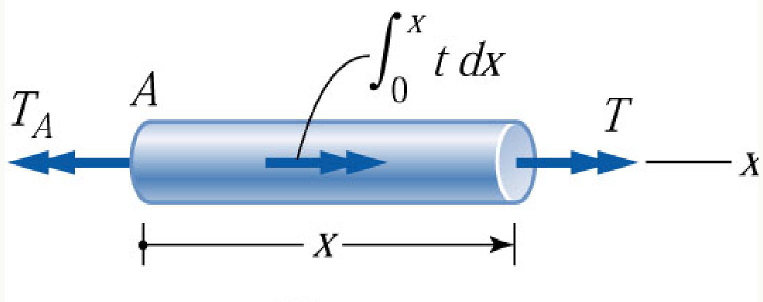
Part 2

The torque T acting on a cross section located at the distance x from the fixed end can be found from the FBD in Fig. (c):

$$\sum M_X = 0 \quad T + \int_0^x t dx - T_A = 0$$

$$T = T_A - \int_0^x t dx = \frac{t_B L}{2} - \int_0^x t_B \frac{x}{L} dx$$

$$= \frac{t_B}{2L}\left(L^2 - x^2\right)$$



(c) FBD

From Eq. (3.4a), the angle θ of twist of the shaft is

$$\theta = \int_0^L \frac{T}{GJ} dx = \frac{t_B}{2LGJ}\int_0^L \left(L^2 - x^2\right) dx = \frac{t_B L^2}{3GJ}$$

$$= \frac{200(1.5)^2}{3\left(80\times10^9\right)(\pi/32)(0.025)^4} = 0.0489 rad = 2.8^\circ \quad Answer$$

3.3 *Torsion of Thin-Walled Tubes*

- Simple approximate formulas are available for thin-walled tubes. Such members are common in construction where light weight is of paramount importance.
- The tube to be prismatic (constant cross section), but the wall thickness *t* is allowed to vary within the cross section. The surface that lies midway between the inner and outer boundaries of the tube is called the ***middle surface***.

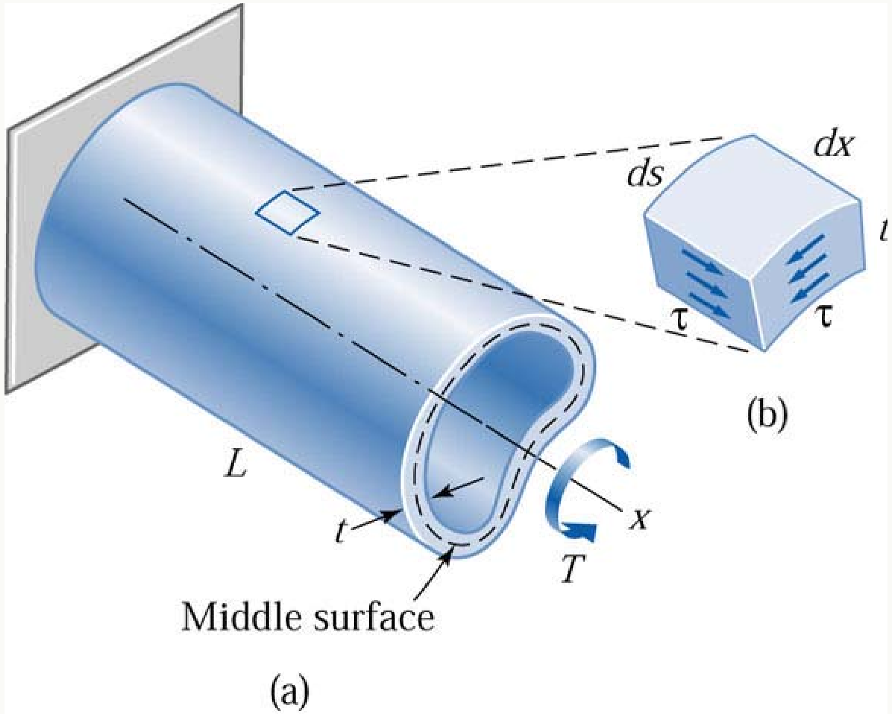
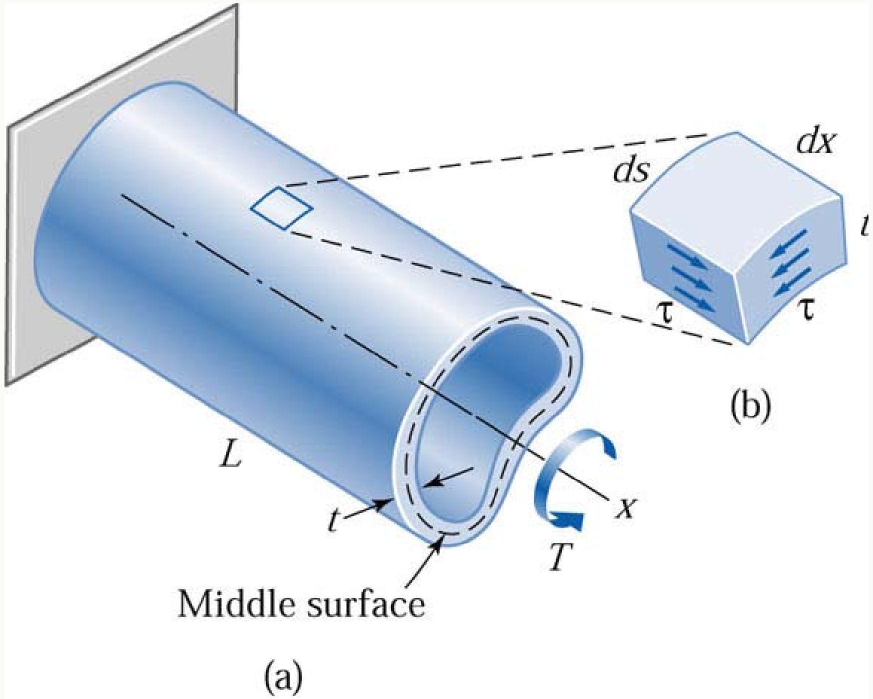


Figure 3.7 (a) Thin-walled tube in torsion; (b) shear stress in the wall of the tube.

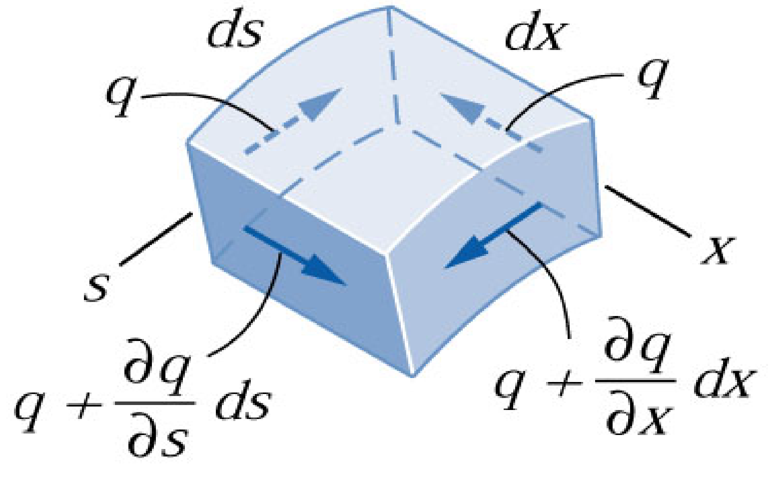
- If thickness t is small compared to the overall dimensions of the cross section, the shear stress τ induced by torsion can be shown to be almost constant through the wall thickness of the tube and directed tangent to the middle surface, in Fig. (3.7b).
- At this time, it is convenient to introduce the concept of ***shear flow q***, defined as the shear force per unit edge length of the middle surface.
- the shear flow q is

$$q = \tau t \qquad (3.7)$$

If the shear stress is not constant through the wall thickness, then τ in Eq. (3.7) should be viewed as the average shear stress.



- *the shear flow is constant throughout the tube*. This result can be obtained by considering equilibrium of the element shown in Fig. 3.7(c).
- In labeling the shear flows, we assume that q varies in the longitudinal (x) as well as the circumferential (s) directions. The force acting on each side of the element is equal to the shear flow multiplied by the edge length, resulting in the equilibrium equations .



(c)

(c) Shear flows on wall element.

$$\sum F_x = 0 \quad \left(q + \frac{\partial q}{\partial s} ds\right) dx - q dx = 0$$

$$\sum F_s = 0 \quad \left(q + \frac{\partial q}{\partial x} dx\right) ds - q ds = 0$$

$\partial q / \partial x = \partial q / \partial s = 0$, thereby proving that the shear flow is constant throughout the tube.

- The shear force is $dP = qds$. The moment of the force about an arbitrary point 0 is $rdP = (qds)r$, where r is the perpendicular distance of *0* from the line of action of dP. The sum of these moment must be equal to the applied torque T; that is,

$$T = \oint_s qrds \qquad \text{(a)}$$

Which the integral is taken over the closed curve formed by the intersection of the middles surface and the cross section, called the ***median line***.

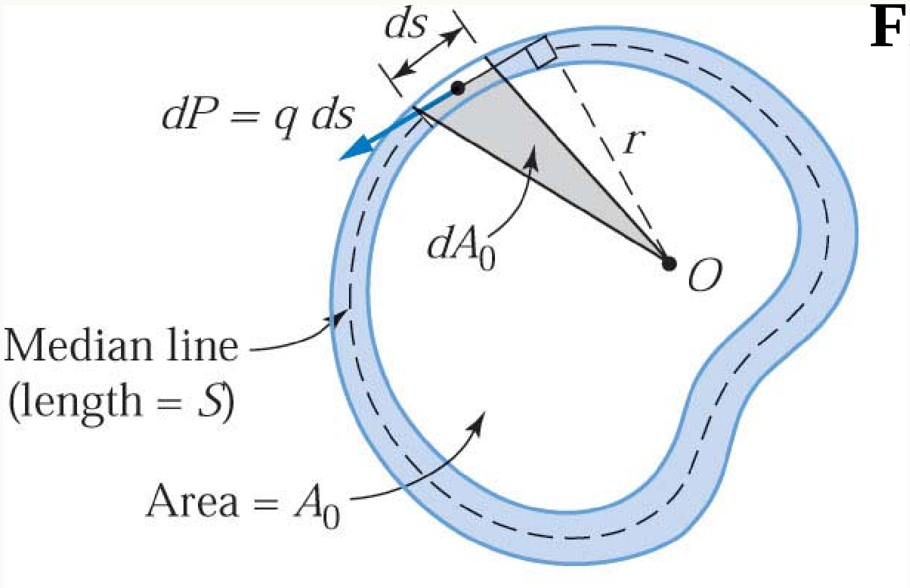
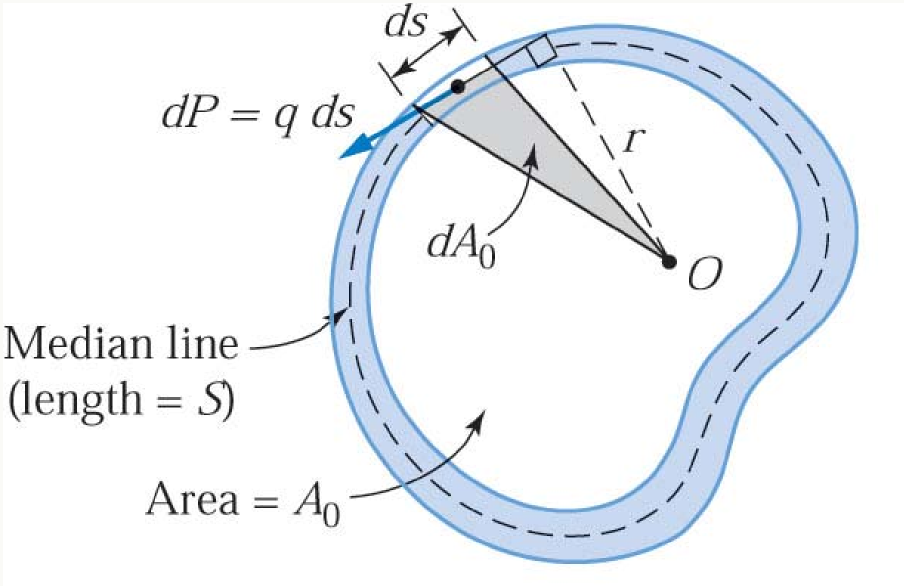


Figure 3.8 Calculating the resultant of the shear flow acting on the cross section of the tube. Resultant of a couple equal to the internal torque *T*.

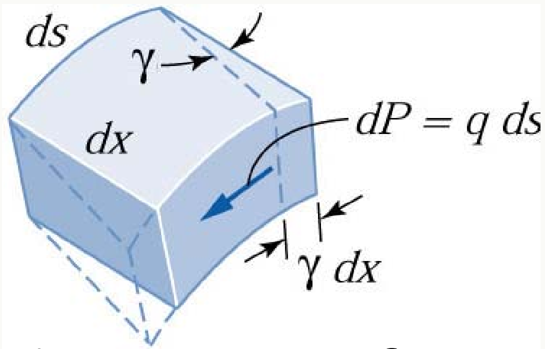
- But from Fig. 3.8 we see that $r\,ds = 2dA_o$, where dA_o is the area of the shaded triangle. Therefore, $\oint_s r\,ds = 2A_0$, where A_o is the area of the cross section that is enclosed by the median line.



$$T = \oint_s q r\,ds \qquad T = 2A_0 q \qquad (3.8a)$$

- from which the shear flow is

$$q = \frac{T}{2A_0} \qquad (3.8b)$$



- Determining the work done by the shear flow acting on the element in Fig. 3.7(c).

$$dU = \frac{1}{2}\left(force \times distance\right) = \frac{1}{2}(q\,ds)(\gamma\,dx)$$

Figure 3.9 Deformation of element caused by shear flow.

Substituting $\gamma = \tau / G = q / Gt$ yields

$$dU = \frac{1}{2}(qds)(\gamma dx) \qquad dU = \frac{q^2}{2Gt} dsdx \tag{b}$$

The work *U* of the shear flow for the entire tube is obtained by integrating Eq. (b) over the middle surface of the tube. Noting that q and *G* are constants and t is independent of *x*,

$$U = \frac{q^2}{2G}\int_0^L \left(\oint_s \frac{ds}{t}\right) dx = \frac{q^2 L}{2G}\oint_s \frac{ds}{t} \tag{c}$$

Conservation of energy requires *U* to be equal to the work of the applied torque that is, $U = T\theta/2$. After substituting the expression for q from Eq. (3.8b) into Eq. (c),

$$q = \frac{T}{2A_0} \qquad \left(\frac{T}{2A_0}\right)^2 \frac{L}{2G}\oint_s \frac{ds}{t} = \frac{1}{2} T\theta$$

The angle of twist of the tube is

$$\theta = \frac{TL}{4GA_0^2} \oint_s \frac{ds}{t} \qquad (3.9a)$$

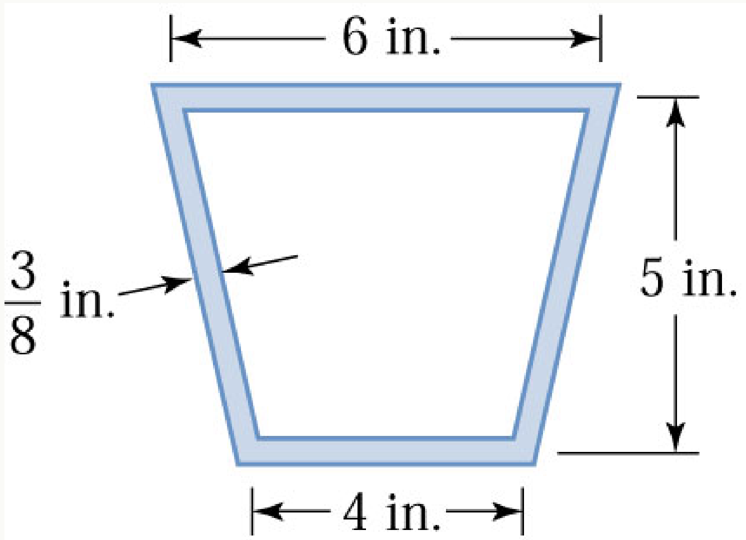
If t is constant, we have $\oint_s (ds/t) = S/t$, where S is the length of the median line Therefore, Eq. (3.9a) becomes

$$\theta = \frac{TLS}{4GA_0^2 t} \quad (\text{ constant } t) \qquad (3.9b)$$

- If the tube is not cylindrical, its cross sections do not remain plane but tend to warp. Tube with very thin walls can fail by bucking which the stresses are still within their elastic ranges. steel tubes of circular cross section require $r/t < 50$ to forestall buckling due to torsion.
- Shape re-entrant corners in the cross section of the tube should also be avoided because they cause stress concentration. The shear stress at the inside boundary of a corner can be considerably higher than the average stress.

***Sample Problem* 3.5**

A steel tube with the cross section shown carries a torque T. The tube is 6 ft long and has a constant wall thickness of 3/8 in. (1) Compute the torsional stiffness $k = T/\theta$ of the tube. (2) If the tube is twisted through 0.5°, determine the shear stress in the wall of the tube. Use G =12 × 106 psi. and neglect stress concentrations at the corners.



3.5 Figure (a)

Solution

Part1

Because the wall thickness is constant, the angle of twist is given by Eq. (3.9b):

$$\theta = \frac{TLS}{4GA_0^2 t}$$

Therefore, the torsional stiffness of the tube can be computed from

$$\kappa = \frac{T}{\theta} = \frac{4GA_0^2 t}{LS}$$

The area enclosed by the median line is

$$A_0 = averagewit\ h \times height = \left(\frac{6+4}{2}\right)(5) = 25 in.^2$$

And the length of the median line is

$$S = 6 + 4 + 2\sqrt{1^2 + 5^2} = 20.20 in.$$

Consequently, the torsional stiffness becomes

$$\kappa = \frac{4(12\times10^6)(25)^2(3/8)}{(6\times12)(20.20)} = 7.735\times10^6\ lb\cdot in./rad$$

$$= 135.0 \times 10^3 \text{ lb} \cdot \text{ in./deg} \qquad \textit{Answer}$$

Part 2

The toque required to produce an angle of twist of 0.5° is

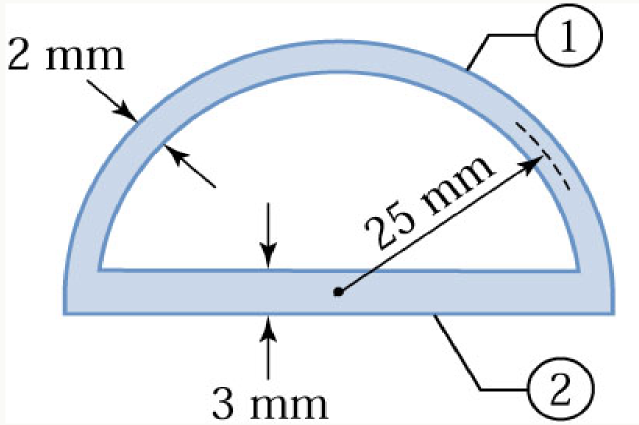
$$T = k\,\theta = (135.0 \times 10^3)(0.5) = 67.5 \times 10^3 \text{ lb} \cdot \text{ in.}$$

which results in the shear flow $q = \frac{T}{2A_0} = \frac{67.5\times10^3}{2(25)} = 1350 lb/in.$

The corresponding shear stress is $\tau = \frac{q}{t} = \frac{1350}{3/8} = 3600\, psi$ Answer

Sample Problem 3.6

An aluminum tube, 1.2 m long, has the semicircular cross section shown in the figure. If stress concentrations at the corners are neglected, determine (1) the torque that causes a maximum shear stress of 40MPa.and (2) the corresponding angle of twist of the tube. Use G = 28 GPa for aluminum.



Solution

the shear flow that causes a maximum shear stress of 40 MPa is

$$q = \tau t = (40 \times 10^6)(0.002) = 80 \times 10^3 \text{ N/m}$$

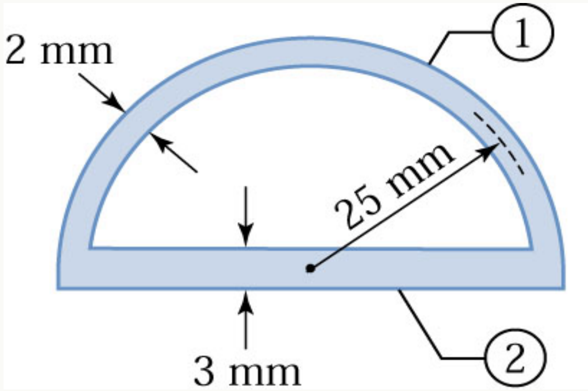
The cross-sectional area enclosed by the median line is

$$A_0 = \frac{\pi r^2}{2} = \frac{\pi (0.25)^2}{2} = 0.9817 \times 10^{-3} m^2$$

$$T = 2A_0 q = 2(0.9817 \times 10^{-3})(80 \times 10^{3}) = 157.07 N \cdot m$$

Part 2

The cross section consists of two parts, labeled 1 and 2 in the figure, each having a constant thickness.



$$\oint_s \frac{ds}{t} = \frac{1}{t_1}\int_{s1} ds + \frac{1}{t_2}\int_{s2} ds = \frac{S_1}{t_1} + \frac{S_2}{t_2}$$

where S_1 and S_2 are the lengths of parts 1 and 2, respectively.

$$\oint_s \frac{ds}{t} = \frac{\pi\gamma}{t_1} + \frac{2\gamma}{t_2} = \frac{\pi(25)}{2} + \frac{2(25)}{3} = 55.94$$

and Eq. (3.9a) yields for the angle of twist

$$\theta = \frac{TL}{4GA_0^2}\oint_s \frac{ds}{t} = \frac{157.07(1.2)}{4(28 \times 10^{9})(09817 \times 10^{-3})^2}(55.94)$$

$$= 0.0977 rad = 5.60^\circ \quad \textit{Answer}$$